Evolution of solutions' support of NPE

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Let $Q_T = (0,T) \times \Omega,\ 0 < T < \infty,\ \Omega \subset \{x \in \mathbb{R}^n : |x| > 1\}$ be a bounded domain in $\mathbb{R}^n,\ n \geqslant 1$, with C^1–boundary $\partial\Omega = \partial_0\Omega \cup \partial_1\Omega$, where $\partial_0\Omega = \{x \in \mathbb{R}^n : |x| = 1\}$, $\partial_1\Omega \subset \{x \in \mathbb{R}^n : |x| > l\}, l = const > 1$. The aim of this brief communication is to investigate the behavior of weak solutions of the following initial-boundary problem:

$$u_t - \sum_{i=1}^{n} \big(a_i(t,x,u,\nabla_x u)\big)_{x_i} + g(t,x)|u|^{q-1}\, u = 0 \ \text{ in } \ Q_T,\ \ 0 < q < 1; \tag{1}$$

$$u(t,x) = f(t,x) \ \text{ on } \ (0,T) \times \partial_0\Omega,\ \ u(t,x) = 0 \ \text{ on } \ (0,T) \times \partial_1\Omega; \tag{2}$$

$$u(0,x) = 0 \qquad \forall x \in \Omega. \tag{3}$$

Here the functions $a_i(t,x,s,\xi)$ $(i = 1,...,n)$ are continuous in all arguments and satisfy the following conditions for $(t,x,s,\xi) \in (0,T) \times \Omega \times \mathbb{R}^1 \times \mathbb{R}^n$: $|a_i(t,x,s,\xi)| \leq d_1|\xi|$, $d_1 = const < \infty$, $\sum\limits_{i=1}^{n} \big(a_i(t,x,s,\xi) - a_i(t,x,s,\eta)\big)(\xi_i - \eta_i) \geq d_0|\xi - \eta|^2$, $d_0 = const > 0$. The absorption potential $g(t,x)$ is continuous nonnegative function such that $g(t,x) > 0$ $\forall (t,x) \in (0,T] \times \overline{\Omega};\ \ g(0,x) = 0\ \forall\, x \in \overline{\Omega}$. Following [1], by weak (or energy) solution of problem (1)–(3) we understand the function $u(t,\cdot) \in f(t,\cdot) + L_2\big(0,T; H^1(\Omega,\partial\Omega)\big)$ such that $u_t(t,\cdot) \in L_2\big(0,T; \big(H^1(\Omega,\partial\Omega)\big)^*\big)$, and u satisfies (2), (3) and the integral identity:

$$\int\limits_{(0,T)} \langle u_t,\ \xi\rangle\ dt\ +\int\limits_{(0,T)\times\Omega} \sum_{i=1}^{n} a_i(t,x,u,\nabla_x u)\xi_{x_i}\, dx\, dt\ +\int\limits_{(0,T)\times\Omega} g(t,x)|u|^{q-1}u\,\xi\, dx\, dt = 0$$

$\forall\ \xi \in L_2(0,T; H^1(\Omega,\partial\Omega))$.

We are interested in a phenomenon called "localization of solutions" for a wide classes of nonlinear parabolic equations with a degenerate absorption potential $g(t,x)$. It is well-known that in case of non-degenerate absorption potential: $g(t,x) \geq c_0 > 0\ \forall (t,x) \in (0,T] \times \overline{\Omega}$, an arbitrary energy solution of the considered problem has the finite-speed propagation property for solution's support: $\zeta(t) := \sup\{|x| : x \in supp\ u(t,\cdot)\} < 1 + c(t)$, where $c(t) \to 0$ as $t \to 0$. In particular, this implies the localization of solution (see, e. g., [2]): $\zeta(t) := \sup\{|x| : x \in supp\ u(t,\cdot)\} < c_1 = c_1(T_1) < l \quad \forall\ t:\ 0 \leq t < T_1 = T_1(l) \leq T$.

For various semi-linear parabolic equations, the localization of solutions' supports were studied by many authors (see [3] and references therein). A. S. Kalashnikov [4] was the first who investigated the localization property for the first initial-boundary problem for a 1-D heat equation. More precisely, he proved that solutions possess weak localization property for t separated from 0: $\sup\{\zeta(t) : 0 < \delta \leq t < T\} < c_1 = c_1(\delta) < \infty \;\forall\, \delta > 0$. On the other hand, following G. I. Barenblatt's conjecture on an initial jump of the free boundary, A. S. Kalashnikov in [4] proved that $\inf\{\zeta(t) : 0 < t < t_*\} \geq c_2 = c_2(t_*) > 0$, if potential $g(t,x) = g_0(t)$ decreases fast enough when $t \to 0$.

The analysis of [4] concerns only the case of strongly degenerating boundary regimes $f(t)$. Also, note that the barrier technique of [4] can be applied only to equations that admit the comparison theorems. Our research involutes arbitrary $f(t)$, which are strongly degenerate, weakly degenerate as well as non-degenerate as $t \to 0$. We found sufficient conditions for the strong localization of solutions (that is continuous propagation of support near to $t = 0$). Note, that these conditions are formulated as a subordination of the boundary regime to the absorption potential. For an arbitrary boundary regime (without any subordination conditions), a certain type of weakened localization is obtained. Under some restriction from below on the degeneration of the potential, the strong localization holds for an arbitrary boundary regime (including regimes that do not satisfy any conditions of subordination).

Thus we would like to present the following results in this brief communication. With boundary regime $f(t,x)$, we associate the function, which will be used in statements:

$$F(t) := \sup_{0\leqslant s\leqslant t} \int\limits_{\Omega} f(s,x)^2 dx + \int\limits_0^t \int\limits_{\Omega} \big(|\nabla_x f|^2 + g(t,x)|f(t,x)|^{q+1}\big) dx\, dt + \int\limits_0^t \int\limits_{\Omega} |f_t(t,x)|^2 dx\, dt.$$

Theorem 1. *Let the absorption potential g from equation (1) have a nonnegative monotonic minorant:*

$$g(t,x) \geq g_0(t) > 0 \;\; \forall\, t > 0, \;\; g_0(0) = 0. \tag{4}$$

Let the function $F(\cdot)$ satisfies the following condition of subordination to g_0 from (4): there exists a function $S = S(t) > 0$ such that

$$F(t)^{\frac{(1-\psi)(1-q)}{2}} < S^2 \Bigg(\int\limits_{\tau}^{t} g_0(t)^{1-\theta} dt \Bigg)^2 + D_1 \int\limits_0^{\tau} g_0(t)^{2(1-\theta)} dt, \quad \forall\, \tau \in (0,t) \tag{5}$$

and

$$tS(t) \longrightarrow 0 \ \ as \ \ t \longrightarrow 0, \tag{6}$$

where l from (2) and

$$0 < \theta := \frac{(q+1)+n(1-q)}{2(q+1)+n(1-q)} < 1, \quad 0 < \psi := \frac{n(1-q)}{2(q+1)+n(1-q)} < \theta < 1.$$

Then an arbitrary energy solution $u(t,x)$ of the problem (1)–(3) possesses the strong localization property and the following upper estimate holds:

$$\zeta(t) \leq 1 + c\,t\,S(t) \quad \forall\, t : 0 < t \leq T.$$

Remark. Also we give several simple conditions that guarantee (5), (6) – see [5].

Theorem 2. *Let the absorption potential g from equation (1) satisfies condition (4).*

Then an arbitrary energy solution $u(t,x)$ to problem (1)–(3) possesses the weakened localization property. That is, there exists $\zeta_1(t) \in C(0,\infty)$ such that

$$\zeta(t) \leq \min(\zeta_1(t), c\,L_1) \quad \textit{for all } t > 0,$$

where $\zeta(\cdot)$ the compactification radius and $L_1 = diam\ \Omega$.

Theorem 3. *Let the absorption potential g from equation (1) have a nonnegative monotonic minorant:*

$$g(t,x) \geq g_\omega(t) := \exp\left(-\frac{\omega(t)}{t}\right) \ \ \forall\, t > 0,$$

where $\omega(t)$ is a nonnegative nondecreasing function such that $\omega(t) \to 0$ as $t \to 0$.

Then an arbitrary energy solution $u(t,x)$ of the problem (1)–(3) possesses the strong localization property and the following upper estimate holds:

$$\zeta(t) \leq 1 + \frac{t}{2} + c_1 \left\{ t\, \ln\big(c_2\, F(t)\big) \ + \ c_3\, t\, \ln t^{-1} \ + \ c_4\, \omega\left(\frac{t}{2}\right) \right\}^{\frac{1}{2}} \qquad \forall \ \ t < T.$$

Our approach is adaptation and combination of a variant of local energy method and an estimate method of Saint–Venant's principle type. These methods are the result of a long evolution of ideas coming from the theory of linear elliptic and parabolic equations. The essence of the energy method consists of special inequalities links different energy norms of solutions. This method was developed and used by J. I. Diaz, L. Veron, S. N. Antontsev,

A. Shishkov, R. Kersner, Y. Belaud (see [2], [6], [7]). The second approach is a technique of parameter's introduction. This method was offered by G.A. Iosif'jan and O.A. Oleinik [8]. Note, that offered combined approach [9] can be applied also to higher order equations.

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