MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

SIMON KUZNETS KHARKIV NATIONAL UNIVERSITY OF ECONOMICS

# ELEMENTARY MATHEMATICS (TRIGONOMETRY AND PRE-CALCULUS)

**Textbook**
**for students**
**of the preparatory department**

**Kharkiv**
**S. Kuznets KhNUE**
**2020**

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E43

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All the themes of the school algebra course are outlined on the basis of the educational program of the secondary school of Ukraine. Methods for solving problems in all themes of algebra are considered in detail and solutions of typical examples are given. Numerous tasks for individual solution are offered.
For international students of the preparatory department.

**UDC 51(075.034)**



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# Contents

# Introduction

This textbook is intended for foreign students of the preparatory department of Simon Kuznets Kharkiv National University of Economics.

In the process of mastering the course of elementary mathematics, students acquire the skills in solving problems envisaged by the school program in mathematics in Ukraine

The purpose of the textbook is to remind and expand the basic information on elementary mathematics in terminology and symbolism adopted in Ukraine.

The textbook covers typical examples on all topics, according to the program of the preparatory department. The presentation of the material is accompanied by a detailed explanation of a large number of typical examples; it also demonstrates skills and techniques for solving problems of medium and high complexity, which are denoted by the "*" sign.

Each topic begins with a brief theoretical introduction and a list of the necessary formulas and contains a sufficient number of well-solved typical examples of various degrees of complexity. At the end of each topic, tasks are given for individual work with the answers placed at the end of the textbook.

This is a concise edition, which presents basic notions, formulas, rules, equations, problems, theorems, and methods on each of the themes in a brief form.

The absence of proof and a concise presentation provide for combining a substantial amount of reference material in a single issue.

The textbook is intended for a wide audience of high school graduates (not specialized in mathematics) as well as graduate and postgraduate students.

The first section of the textbook contains information on trigonometric functions, properties of trigonometric functions, basic trigonometric identities, reduction formulas, trigonometric equations and systems of trigonometric equations.

The second section of the textbook is devoted to logarithmic functions, logarithmic equations, systems of exponential and logarithmic equations, exponential inequalities and logarithmic inequalities.

The third section of the textbook deals with limits and derivatives of functions, the differentiation rules, the table of derivatives, investigation of functions.

The fourth section of the textbook concerns antiderivative functions and indefinite integrals, calculation of the area of a figure.

A compact and clear presentation of the material allows the reader to get quick help on (or revise) the desired topic. Special attention is paid to the issues that many high school graduates and students may find difficult to understand.

When selecting the material, the authors have given a pronounced preference to practical aspects; namely, to formulas, problems, methods, and laws that most frequently occur in economics and university education.

For the convenience of a wider audience with different mathematical backgrounds, the authors tried to avoid special terminology whenever possible.

Therefore, some of the topics and methods are outlined in a schematic and somewhat simplified manner, which is sufficient for them to be used successfully in most cases. Many sections were written so that they could be read independently. The material within subsections is arranged in increasing order of complexity. This allows the reader to get to the heart of the matter quickly.

The material of the textbook can be roughly categorized into the following two groups according to meaning:

1. The main text containing a concise, coherent survey of the most important definitions, formulas, equations, methods, and theorems.

2. For the reader's better understanding of the topics and methods under study, numerous examples are given throughout the book.

For the reader's convenience, several mathematical tables, indefinite and definite integrals are presented.

At the end of each subsection, a list of tasks for individual work on the topics of interest to the reader are offered.

The material is selected in such a way that the course of Higher Mathematics is sufficient for studying it.

# 1. The trigonometric functions

## 1.1. The definition of trigonometric functions

1. A right triangle ABC is given (Fig. 1.1).

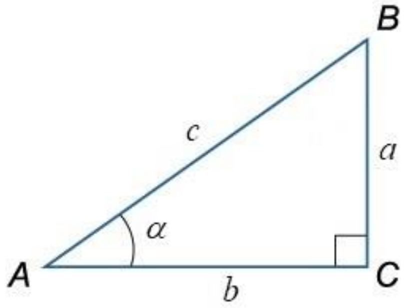



Fig. 1.1. **The right triangle ABC**

The sine of the acute angle $\alpha$ in the right triangle is called the ratio of the opposite leg $a$ to the hypotenuse $c$:

$$\sin\alpha = \frac{a}{c}.$$

The cosine of the acute angle $\alpha$ in the right triangle is called the ratio of the adjacent leg $b$ to the hypotenuse $c$:

$$\cos\alpha = \frac{b}{c}.$$

The tangent of the acute angle $\alpha$ in the right triangle is called the ratio of the opposite leg $a$ to the adjacent leg $b$:

$$\operatorname{tg}\alpha = \frac{a}{b}.$$

The cotangent of the acute angle $\alpha$ in the right triangle is called the ratio of the adjacent leg $b$ to the opposite leg $a$:

$$\operatorname{ctg}\alpha = \frac{b}{a}.$$

2. A circle with a radius $R$ is given (Fig. 1.2).

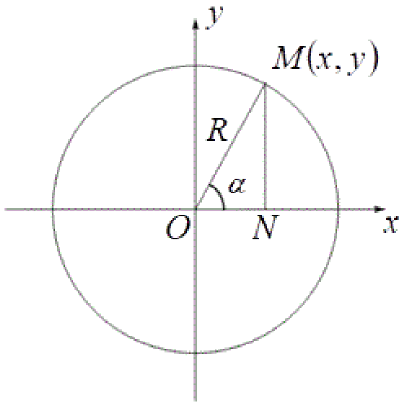



Fig. 1.2. **The circle of the radius $R$**

Let's consider the circle of the radius $R$ with a center in the origin. Let's mark an arbitrary point $M$ with coordinates $(x, y)$ on this circle and connect it with the origin. The segment $OM$ is called a moving radius of the point $M(x, y)$. Let it form an angle $\alpha$ with the axis $Ox$. Different angles (positive and negative) can be formed with the rotation of the radius $OM$.

Using $\Delta OMN$ (Fig. 1.2) let's write the values of the trigonometric functions of the arbitrary angle $\alpha$:

$$\sin\alpha = \frac{y}{R}; \qquad \cos\alpha = \frac{x}{R}; \qquad \operatorname{tg}\alpha = \frac{y}{x}; \qquad \operatorname{ctg}\alpha = \frac{x}{y}.$$

3. A unit circle is given.

Since the values $\sin\alpha$, $\cos\alpha$, $\operatorname{tg}\alpha$ and $\operatorname{ctg}\alpha$ don't depend on the length of the moving radius $OM$, then let's take it as equal to one, i.e. $R = 1$. In this case the circle is called a unit circle. Then the values of the trigonometric functions are defined in the following way.

The sine of the angle $\alpha$ is called the ordinate of the point $M(x, y)$ of the unit circle:

$$\sin\alpha = y.$$

The cosine of the angle $\alpha$ is called the abscissa of the point $M(x, y)$ of the unit circle:

$$\cos\alpha = x.$$

The tangent and the cotangent of the angle $\alpha$ are defined by the relationships:

$$\operatorname{tg}\alpha=\frac{\sin\alpha}{\cos\alpha}\ (\cos\alpha\neq 0),\qquad \operatorname{ctg}\alpha=\frac{\cos\alpha}{\sin\alpha}\ (\sin\alpha\neq 0).$$

## 1.2. The values of trigonometric functions of basic angles

Due to the introduced definitions of trigonometric functions it is possible to calculate the values of the basic angles (Table 1.1).

Table 1.1

**The values of trigonometric functions for basic angles**

| Angle $\alpha$ / Function | $0^\circ$ | $\frac{\pi}{6}\left(30^\circ\right)$ | $\frac{\pi}{4}\left(45^\circ\right)$ | $\frac{\pi}{3}\left(60^\circ\right)$ | $\frac{\pi}{2}\left(90^\circ\right)$ | $\pi\left(180^\circ\right)$ | $\frac{3\pi}{2}\left(270^\circ\right)$ | $2\pi\left(360^\circ\right)$ |
|---|---|---|---|---|---|---|---|---|
| $\sin\alpha$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 | 0 | $-1$ | 0 |
| $\cos\alpha$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | $-1$ | 0 | 1 |
| $\operatorname{tg}\alpha$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{2}$ | – | 0 | – | 0 |
| $\operatorname{ctg}\alpha$ | – | $\sqrt{3}$ | 1 | $\frac{1}{\sqrt{3}}$ | 0 | – | 0 | – |

**Example 1.1.** Calculate: $A=\dfrac{4-2\operatorname{tg}^2 45^\circ+\operatorname{ctg}^2 60^\circ}{3\sin^2 90^\circ-4\cos^2 60^\circ+\operatorname{ctg}45^\circ}$.

*Solution*.

$$A=\frac{4-2\cdot 1+\left(\dfrac{1}{\sqrt{3}}\right)^2}{3\cdot 1-4\cdot\left(\dfrac{1}{2}\right)^2+1}=\frac{4-2+\dfrac{1}{3}}{3-1+1}=\frac{7}{9}.$$

**Example 1.2.** Simplify the expression:

$$A = \frac{\left(a \sin \frac{\pi}{2}\right)^3 + \operatorname{ctg} \frac{3\pi}{2} + (b \cos \pi)^3}{\left(a \cdot \cos 0^\circ\right)^2 - ab \cdot \sin \frac{3\pi}{2} + b^2 + \operatorname{tg} \pi}.$$

*Solution*.

$$A = \frac{a^3 - b^3}{a^2 + ab + b^2 + b^2} = \frac{(a-b)\cdot\left(a^2 + ab + b^2 + b^2\right)}{a^2 + ab + b^2 + b^2} = a - b.$$

## Tasks for individual work

**Task 1.3.** Calculate the values of the expressions:

1) $2\sin\frac{\pi}{3} + 2\cos\frac{\pi}{4} - 3\operatorname{tg}\frac{\pi}{3} + \operatorname{ctg}\frac{\pi}{6}$;

2) $3 - \sin^2\frac{\pi}{2} + 2\cos^2\frac{\pi}{3} - 3\operatorname{tg}\frac{\pi}{4}$;

3) $\dfrac{1 - \cos^2\frac{\pi}{4} + \operatorname{tg}^2\frac{\pi}{3}}{1 + \cos^2\frac{\pi}{4} - \operatorname{tg}^2\frac{\pi}{3}}$.

**Task 1.4.** Simplify the expressions:

1) $3a^2 \sin 2\pi + b^2 \operatorname{tg} 0 - 2ab\cos\pi + b^2 \sin\pi$;

2) $4a^2 \sin^4\frac{\pi}{4} - 6ab \cdot \operatorname{tg}^2\frac{\pi}{6} + \left(b \cdot \operatorname{ctg}\frac{\pi}{4}\right)^2$;

3) $\dfrac{\left(2a \cdot \cos 60^\circ\right)^2 - \left(b \cdot \operatorname{ctg} 45^\circ\right)^2 + \left(3ab \sin 0^\circ\right)^2}{\left(5a \cos 90^\circ\right)^2 + 2a \sin 30^\circ - 2b\cos^2 45^\circ}$.

**Task 1.5.** Plot the angles if:

1) $\cos\alpha = \frac{2}{3}$; 2) $\cos\alpha = -0.4$;

3) $\sin\alpha = 0.25$; 4) $\sin\alpha = -0.5$.

## 1.3. The properties of trigonometric functions

1. The signs of trigonometric functions (Fig. 1.3).

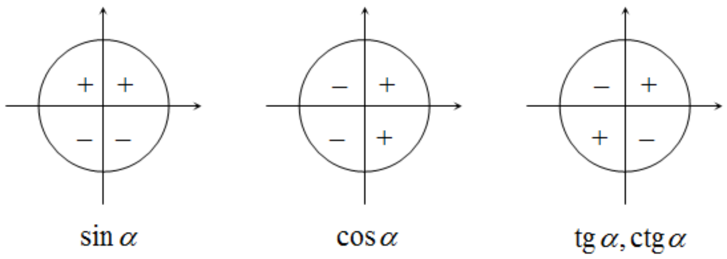



Fig. 1.3. **The signs of trigonometric functions**

For example, $\sin 300^\circ < 0$, because the angle $\alpha = 300^\circ \in 4\text{th}$ quarter and $\cos 300^\circ > 0$.

2. Evenness and oddness of trigonometric functions.

The introduced function cosine is even, but the sine, the tangent and the cotangent are odd:

$$\cos(-\alpha) = \cos\alpha, \quad \sin(-\alpha) = -\sin\alpha, \quad \mathrm{tg}(-\alpha) = -\mathrm{tg}\,\alpha, \quad \mathrm{ctg}(-\alpha) = -\mathrm{ctg}\,\alpha.$$

**Example 1.6.** Calculate the expression:

$$A = \sin\left(-\frac{\pi}{6}\right) - 2\,\mathrm{tg}\left(-\frac{\pi}{4}\right) + \cos\left(-\frac{\pi}{3}\right) - \mathrm{ctg}\left(-\frac{\pi}{2}\right).$$

*Solution*.

$$A = -\sin\left(\frac{\pi}{6}\right) + 2\,\mathrm{tg}\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{3}\right) + \mathrm{ctg}\left(\frac{\pi}{2}\right) = -\frac{1}{2} + 2\cdot 1 + \frac{1}{2} + 0 = 2.$$

**Example 1.7.** Calculate the expression:

$$A = 4\sin^2\left(-\frac{\pi}{3}\right) + 9\,\mathrm{tg}^2\left(-\frac{\pi}{6}\right) - 4\cos^2\left(-\frac{\pi}{3}\right) + 16\,\mathrm{ctg}^2\left(-\frac{\pi}{4}\right).$$

*Solution*.

$$A = 4\left(-\sin\frac{\pi}{3}\right)^2 + 9\left(-\mathrm{tg}\frac{\pi}{6}\right)^2 - 4\cos^2\frac{\pi}{3} + 16\left(-\mathrm{ctg}\frac{\pi}{4}\right)^2 =$$

$$=4\sin^2\left(\frac{\pi}{3}\right)+9\,\mathrm{tg}^2\left(\frac{\pi}{6}\right)-4\cos^2\left(\frac{\pi}{3}\right)+16\,\mathrm{ctg}^2\left(\frac{\pi}{4}\right)=$$

$$=4\left(\frac{\sqrt{3}}{2}\right)^2+9\cdot\left(\frac{1}{\sqrt{3}}\right)^2-4\cdot\left(\frac{1}{2}\right)+16\cdot 1=3+3-1+16=21.$$

3. Periodicity of trigonometric functions.

Let's remember that the function $y=f(x)$ is called periodic with the period $T\neq 0$ if for any value $x$ of the domain of the function definition the following equalities are fulfilled:

$$f(x+T)=f(x-T)=f(x).$$

It follows from the definition of trigonometric functions on a unit circle that the following functions are periodic:

$$\sin\alpha=\sin(\alpha\pm 2\pi)=\sin(\alpha\pm 4\pi)=...=\sin(\alpha\pm 2\pi\cdot k),\ k\in Z;$$
$$\cos\alpha=\cos(\alpha\pm 2\pi)=\cos(\alpha\pm 4\pi)=...=\cos(\alpha\pm 2\pi\cdot k),\ k\in Z;$$
$$\mathrm{tg}\alpha=\mathrm{tg}(\alpha\pm\pi)=\mathrm{tg}(\alpha\pm 2\pi)=...=\mathrm{tg}(\alpha\pm\pi\cdot k),\ k\in Z;$$
$$\mathrm{ctg}\,\alpha=\mathrm{ctg}(\alpha\pm\pi)=\mathrm{ctg}(\alpha\pm 2\pi)=...=\mathrm{ctg}(\alpha\pm\pi\cdot k),\ k\in Z.$$

We can see that the smallest positive period of a sine and a cosine equals $2\pi$; for a tangent and a cotangent it is $\pi$.

For example, a) $\sin 1950^\circ=\sin(150^\circ+1800^\circ)=\sin(150^\circ+360^\circ\cdot 5)=$ $=\sin 150^\circ.$

b) $\cos(-315^\circ)=\cos(315^\circ)=\cos(360^\circ-45^\circ)=\cos(-45^\circ)=\cos(45^\circ)=\frac{\sqrt{2}}{2}.$

4. Boundedness of trigonometric functions.

It follows from the definition of the sine and the cosine that:

$$-1\leq\sin\alpha\leq 1,\quad -1\leq\cos\alpha\leq 1 \qquad \text{or} \qquad |\sin\alpha|\leq 1,\quad |\cos\alpha|\leq 1.$$

It follows from the definition of the tangent and the cotangent that:

$$-\infty \le \operatorname{tg}\alpha \le \infty \text{ if } \cos\alpha \ne 0;$$
$$-\infty \le \operatorname{ctg}\alpha \le \infty \text{ if } \sin\alpha \ne 0.$$

## Tasks for individual work

**Task 1.8.** Define the signs of the following values:

1) $\sin 170^\circ$; 2) $\cos 300^\circ$; 3) $\operatorname{tg} 160^\circ$; 4) $\operatorname{ctg} 315^\circ$;

5) $\operatorname{tg} 450^\circ$; 6) $\sin 400^\circ$; 7) $\sin\dfrac{7\pi}{3}$; 8) $\cos\dfrac{4\pi}{3}$.

**Task 1.9.** Define the signs of the products:

1) $\sin 170^\circ \cdot \sin 120^\circ$; 2) $\cos 210^\circ \cdot \sin 210^\circ$;

3) $\cos 200^\circ \cdot \sin 110^\circ$; 4) $\operatorname{tg} 140^\circ \cdot \operatorname{tg} 220^\circ$;

5) $\sin 150^\circ \cdot \cos 150^\circ \cdot \operatorname{tg} 150^\circ$; 6) $\sin 320^\circ \cdot \cos 125^\circ \cdot \operatorname{tg} 250^\circ$.

**Task 1.10.** Calculate the expressions:

1) $\cos(-\pi)\cdot\cos\left(-\dfrac{\pi}{2}\right)\cdot\sin\left(-\dfrac{3\pi}{2}\right)$

2) $\cos^2\left(-\dfrac{\pi}{3}\right)-\operatorname{ctg}^2\left(-\dfrac{\pi}{6}\right)+\sin^2\left(-\dfrac{\pi}{6}\right)$;

3) $\dfrac{2-\sin^2\left(-\dfrac{\pi}{6}\right)-\cos^2\left(-\dfrac{\pi}{6}\right)}{1-2\cos^2\left(-\dfrac{\pi}{6}\right)}$;

4) $2\sin\left(-\dfrac{\pi}{6}\right)+3\cos\left(-\dfrac{\pi}{2}\right)-3\operatorname{ctg}\left(-\dfrac{\pi}{4}\right)+4\operatorname{tg}0$.

**Task 1.11.** Simplify the expressions:

1) $\dfrac{\left(a\sin\left(-\dfrac{\pi}{2}\right)\right)^4-\left(b\cdot\operatorname{tg}\left(-\dfrac{\pi}{4}\right)\right)^4}{(a\cos 2\pi)^2-\left(b\sin\left(-\dfrac{\pi}{2}\right)\right)^2}$;

2) $\left(-a\sin\left(-\frac{3\pi}{2}\right)\right)^3+(ab\cdot\mathrm{tg}(2\pi))^3+(b\cos 0)^3$;

3) $\dfrac{(a\cos 0)^4-\left(b\cdot\mathrm{tg}\left(-\frac{\pi}{4}\right)\right)^4}{2a^2\cos\left(-\frac{\pi}{3}\right)+2ab\cos(-\pi)-b^2\,\mathrm{tg}\left(-\frac{\pi}{4}\right)}$.

**Task 1.12.** Define the sign of the expression if $0<\alpha<\frac{\pi}{2}$.

1) $\sin\left(\frac{\pi}{2}+\alpha\right)$; 2) $\cos\left(\frac{3\pi}{2}-\alpha\right)$; 3) $\sin(3\pi+\alpha)$;

4) $\sin(2\pi-\alpha)$; 5) $\cos(2\pi+\alpha)$; 6) $\mathrm{tg}\left(\frac{3\pi}{2}+\alpha\right)$;

7) $\cos\left(\frac{\pi}{2}-\alpha\right)\cdot\mathrm{tg}(2\pi-\alpha)\cdot\mathrm{ctg}\left(\frac{\pi}{2}+\alpha\right)$;

8) $\sin\left(\frac{3\pi}{2}+\alpha\right)\cdot\cos(\pi+\alpha)\cdot\mathrm{tg}(\pi+\alpha)$;

9) $\cos^2(\pi+\alpha)\cdot\mathrm{tg}^2\left(\frac{\pi}{2}+\alpha\right)\cdot\mathrm{ctg}^2(\pi-\alpha)$.

## 1.4. The graphs of trigonometric functions

1. The function $y=\sin x$ is given.

The basic properties are:

a) $x\in(-\infty;\infty)$; b) $y\in[-1;1]$;

c) it's an odd function: $\sin(-x)=-\sin(x)$;

d) it's a periodic function with the period $T=2\pi$.

The graph of the function $y=\sin x$ is given in Fig. 1.4.

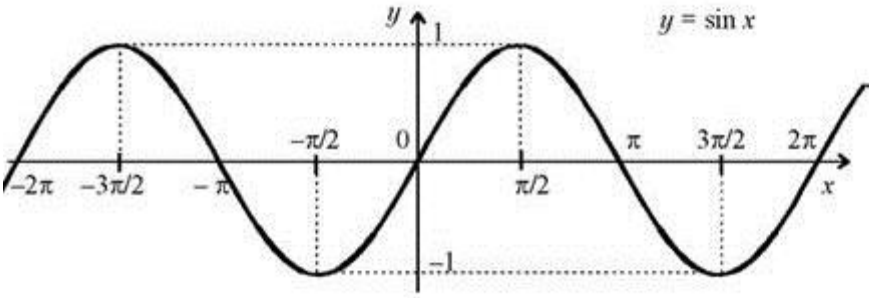



Fig. 1.4. **The graph of the function** $y = \sin x$

2. The function $y = \cos x$ is given.

The basic properties are:

a) $x \in (-\infty;\infty)$; b) $y \in [-1;1]$;

c) it's an even function: $\cos(-x) = \cos(x)$;

d) it's a periodic function with the period $T = 2\pi$.

The graph of the function $y = \cos x$ is given in Fig. 1.5.

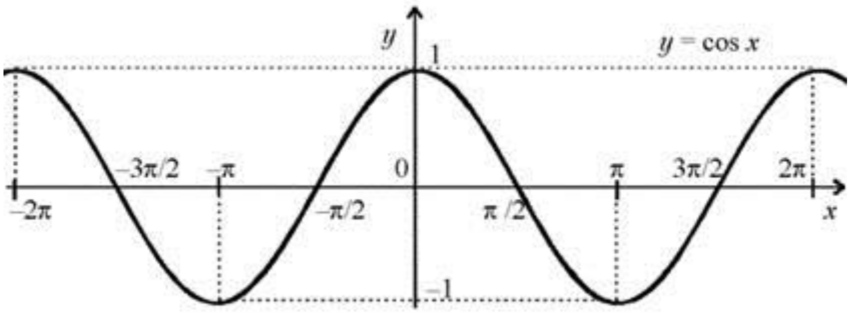



Fig. 1.5. **The graph of the function** $y = \cos x$

3. The function $y = \text{tg}\, x$ is given.

The basic properties are:

a) $x \neq \dfrac{\pi}{2} + \pi n, n \in Z$ ; b) $y \in (-\infty;\infty)$;

c) it's an odd function: $\text{tg}(-x) = -\text{tg}(x)$;

d) it's a periodic function with the period $T = \pi$.

The graph of the function $y = \text{tg } x$ is given in Fig. 1.6.

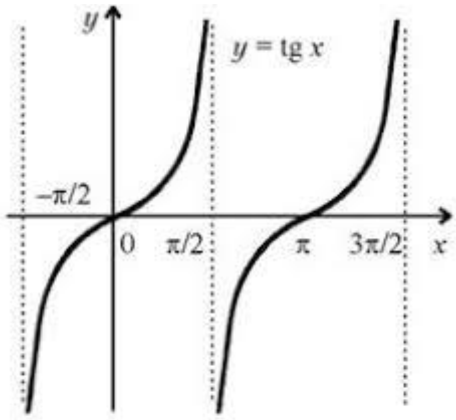



Fig. 1.6. **The graph of the function** $y = \text{tg}\, x$

4. The function $y = \text{ctg}\, x$ is given.

The basic properties are:

a) $x \neq \pi n, n \in Z$; b) $y \in (-\infty; \infty)$;

c) it's an odd function $\text{ctg}(-x) = -\text{ctg}(x)$;

d) it's a periodic function with the period $T = \pi$.

The graph of the function $y = \text{ctg } x$ is given in Fig. 1.7.

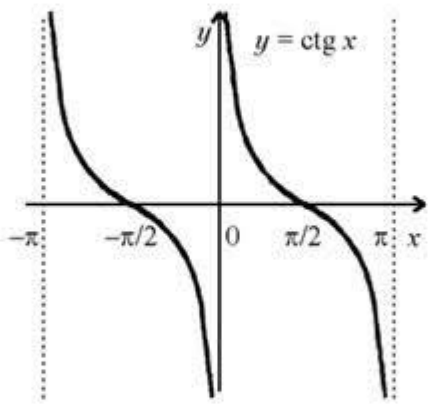



Fig. 1.7. **The graph of the function** $y = \text{ctg}\, x$

## Tasks for individual work

**Task 1.13.** Draw the graphs of the functions:

1) $y = |\sin x|$; 2) $y = \left(\sqrt{\sin x}\right)^2$; 3) $y = \dfrac{|\sin x|}{\sin x}$;

4) $y=\sin x+|\sin x|$; 5) $y=|\cos x|$; 6) $y=\cos x-|\cos x|$;

7) $y=\left(\sqrt{\cos x}\right)^2$; 8) $y=\dfrac{|\cos x|}{\cos x}$; 9) $y=|\operatorname{tg} x|$; 10) $y=|\operatorname{ctg} x|$;

11) $y=\operatorname{tg} x\cdot\operatorname{ctg} x$; 12) $y=\operatorname{tg} x+|\operatorname{tg} x|$; 13) $y=\operatorname{ctg} x+|\operatorname{ctg} x|$.

## 1.5. Basic trigonometric identities

A trigonometric identity is an equality which includes trigonometric functions. It is satisfied with the help of arbitrary valid values of an angle or argument of trigonometric functions.

The basic trigonometric identities are:

$$\sin^2\alpha+\cos^2\alpha=1; \tag{1.1}$$

$$\operatorname{ctg}\alpha=\frac{\cos\alpha}{\sin\alpha};\ (\sin\alpha\neq 0); \qquad \operatorname{tg}\alpha=\frac{\sin\alpha}{\cos\alpha};\ (\cos\alpha\neq 0) \tag{1.2}$$

$$\operatorname{tg}\alpha\cdot\operatorname{ctg}\alpha=1; \tag{1.3}$$

$$1+\operatorname{tg}^2\alpha=\frac{1}{\cos^2\alpha}; \qquad 1+\operatorname{ctg}^2\alpha=\frac{1}{\sin^2\alpha}. \tag{1.4}$$

Using these identities we can express other functions through a given trigonometric function.

$$\cos^2\alpha=1-\sin^2\alpha; \qquad \cos\alpha=\pm\sqrt{1-\sin^2\alpha}; \tag{1.5}$$

$$\sin^2\alpha=1-\cos^2\alpha; \qquad \sin\alpha=\pm\sqrt{1-\cos^2\alpha}; \tag{1.6}$$

$$\operatorname{tg}\alpha=\frac{1}{\operatorname{ctg}\alpha}; \qquad \operatorname{ctg}\alpha=\frac{1}{\operatorname{tg}\alpha}; \tag{1.7}$$

$$\cos^2\alpha=\frac{1}{1+\operatorname{tg}^2\alpha}; \qquad \sin^2\alpha=\frac{\operatorname{tg}^2\alpha}{1+\operatorname{tg}^2\alpha}; \tag{1.8}$$

**Example 1.14.** It is known that $\cos\alpha = -\frac{12}{13}, \quad \alpha \in \left(\pi; \frac{3\pi}{2}\right)$.

Find: 1) $\sin\alpha$; 2) $\text{tg}\,\alpha$; 3) $\text{ctg}\,\alpha$.

*Solution*.

1) Using formula (1.5) and considering this condition (the angle α belongs to the 3rd quarter, where $\sin\alpha < 0$, we have:

$$\sin\alpha = -\sqrt{1-\cos^2\alpha} = -\sqrt{1-\left(-\frac{12}{13}\right)^2} = -\sqrt{\frac{25}{169}} = -\frac{5}{13}.$$

2) $\text{tg}\,\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{5}{13} : \left(-\frac{12}{13}\right) = -\frac{5}{12}.$

3) $\text{ctg}\,\alpha = \frac{1}{\text{tg}\,\alpha} = -\frac{12}{5}.$

**Example 1.15.** It is known that $\text{tg}\,\alpha = -\frac{3}{4}, \quad \alpha \in \left(\frac{\pi}{2}; \pi\right)$.

Find: 1) $\text{ctg}\,\alpha$; 2) $\cos\alpha$; 3) $\sin\alpha$.

*Solution*. 1) $\text{ctg}\,\alpha = \frac{1}{\text{tg}\,\alpha} = \frac{1}{-3/4} = -\frac{4}{3};$

2) using formula (1.6) we get $\cos^2\alpha = \frac{1}{1+\text{tg}^2\alpha} = \frac{1}{1+\frac{9}{16}} = \frac{16}{25};$ then

$\cos\alpha = -\sqrt{\frac{16}{25}} = -\frac{4}{5};$ (according to the condition we have $\cos\alpha < 0$, because $\alpha \in \left(\frac{\pi}{2}; \pi\right)$.

3) $\sin^2\alpha = 1-\cos^2\alpha = 1-\frac{16}{25} = \frac{9}{25}; \quad \sin\alpha = +\sqrt{\frac{9}{25}} = \frac{3}{5}.$

**Example 1.16.** Calculate $\frac{3\sin\alpha + 4\cos\alpha}{\cos\alpha - \sin\alpha}$, if $\text{tg}\,\alpha = -\frac{3}{4}$.

*Solution*. Let's transform the given fraction with the help of the division of the numerator and the denominator by $\cos\alpha$.

$$A=\frac{3\sin\alpha+4\cos\alpha}{\cos\alpha-\sin\alpha}=\frac{3\dfrac{\sin\alpha}{\cos\alpha}+4\dfrac{\cos\alpha}{\cos\alpha}}{\dfrac{\cos\alpha}{\cos\alpha}-\dfrac{\sin\alpha}{\cos\alpha}}=\frac{3\operatorname{tg}\alpha+4}{1-\operatorname{tg}\alpha}.$$

At $\operatorname{tg}\alpha=-\frac{1}{3}$ $\quad A=\dfrac{3\cdot\left(-\frac{1}{3}\right)+4}{1+\frac{1}{3}}=\dfrac{9}{4}.$

**Example 1.17.** Prove the identity $\dfrac{1+2\sin\alpha\cdot\cos\alpha}{\sin^2\alpha-\cos^2\alpha}=\dfrac{\operatorname{tg}\alpha+1}{\operatorname{tg}\alpha-1}$.

*Solution*. Let's transform the left part of the identity:

$$\frac{1+2\sin\alpha\cdot\cos\alpha}{\sin^2\alpha-\cos^2\alpha}=\frac{\sin^2\alpha+\cos^2\alpha+2\sin\alpha\cdot\cos\alpha}{\sin^2\alpha-\cos^2\alpha}=$$

$$=\frac{(\sin\alpha+\cos\alpha)^2}{(\sin\alpha-\cos\alpha)(\sin\alpha+\cos\alpha)}=\frac{(\sin\alpha+\cos\alpha)}{(\sin\alpha-\cos\alpha)}=$$

$$=\frac{(\sin\alpha+\cos\alpha):\cos\alpha}{(\sin\alpha-\cos\alpha):\cos\alpha}=\frac{\operatorname{tg}\alpha+1}{\operatorname{tg}\alpha-1},$$

which was to be proved.

## Tasks for individual work

**Task 1.18.** Calculate the values of the rest three trigonometric functions, if

1) $\sin\alpha=-\dfrac{5}{13},\quad \alpha\in\left(\dfrac{3\pi}{2},2\pi\right);$

2) $\cos\alpha = -\frac{8}{17}, \quad \alpha \in \left(\frac{\pi}{2}, \pi\right);$

3) $\mathrm{tg}\,\alpha = \frac{5}{13}, \quad \alpha \in \left(\pi, \frac{3\pi}{2}\right);$

4) $\mathrm{ctg}\,\alpha = -\frac{7}{24}, \quad \alpha \in \left(\frac{3\pi}{2}, 2\pi\right).$

**Task 1.19.** Calculate $\frac{\sin\alpha + \mathrm{tg}\,\alpha}{1+\cos\alpha}$, if $\sin\alpha = -\frac{5}{13}, \quad \alpha \in \left(-\frac{\pi}{2}, 0\right).$

**Task 1.20.** Calculate $\frac{3\sin\alpha \cdot \cos\alpha}{2\sin^2\alpha - 3\cos^2\alpha}$, if $\mathrm{ctg}\,\alpha = -2.$

**Task 1.21.** Calculate $\frac{\mathrm{tg}\,\alpha + \mathrm{ctg}\,\alpha}{1+\mathrm{tg}^2\alpha}$, if $\mathrm{tg}\,\alpha = \sqrt{3}.$

**Task 1.22.** Calculate $\frac{3\sin^2 x - 3\cos^2 x}{2\sin^2 x + 3\cos^2 x}$, if $\mathrm{tg}\,x = 3.$

**Task 1.23.** Simplify:

1) $\sin^2\alpha + \mathrm{tg}^2\alpha + \cos^2\alpha;$ 2) $\sin^4\alpha - \cos^4\alpha + \cos^2\alpha;$

3) $(\sin\alpha + \cos\alpha)^2 + (\sin\alpha - \cos\alpha)^2;$ 4) $\frac{\sin\alpha}{1+\cos\alpha} + \frac{\sin\alpha}{1-\cos\alpha};$

5) $\mathrm{tg}\,\alpha + \frac{\cos\alpha}{1+\sin\alpha};$ 6) $\sqrt{\frac{1-\sin\alpha}{1+\sin\alpha}} + \sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}}.$

**Task 1.24.** Prove the identities:

1) $\sin^4\alpha + \sin^2\alpha \cdot \cos^2\alpha + \cos^2\alpha = 1;$

2) $\sqrt{1+2\sin\alpha \cdot \cos\alpha} = |\cos\alpha + \sin\alpha|;$

3) $\mathrm{ctg}\,\alpha + \frac{\sin\alpha}{1+\cos\alpha} = \frac{1}{\sin\alpha};$

4) $\sin^2\alpha(1+\mathrm{ctg}\,\alpha) + \cos^2\alpha(1+\mathrm{tg}\,\alpha) = \sin\alpha + \cos\alpha;$

5) $\frac{1-(\sin\alpha + \cos\alpha)^2}{\sin\alpha \cdot \cos\alpha - ctg\alpha} = 2tg^2\alpha;$

6) $\frac{\cos^3\alpha - \sin^3\alpha}{1+\sin\alpha \cdot \cos\alpha} = \cos\alpha - \sin\alpha.$

## 1.6. Theorems of addition

For finding the trigonometric functions of a sum and a difference of two arguments the following formulas are used:

$$\sin(\alpha+\beta)=\sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta; \tag{1.9}$$

$$\sin(\alpha-\beta)=\sin\alpha\cdot\cos\beta-\cos\alpha\cdot\sin\beta; \tag{1.10}$$

$$\cos(\alpha+\beta)=\cos\alpha\cdot\cos\beta-\sin\alpha\cdot\sin\beta; \tag{1.11}$$

$$\cos(\alpha-\beta)=\cos\alpha\cdot\cos\beta+\sin\alpha\cdot\sin\beta; \tag{1.12}$$

$$\mathrm{tg}(\alpha+\beta)=\frac{\mathrm{tg}\,\alpha+\mathrm{tg}\,\beta}{1-\mathrm{tg}\,\alpha\cdot\mathrm{tg}\,\beta}; \tag{1.13}$$

$$\mathrm{tg}(\alpha-\beta)=\frac{\mathrm{tg}\,\alpha-\mathrm{tg}\,\beta}{1+\mathrm{tg}\,\alpha\cdot\mathrm{tg}\,\beta}; \tag{1.14}$$

$$\mathrm{ctg}(\alpha\pm\beta)=\frac{\mathrm{ctg}\,\alpha\cdot\mathrm{ctg}\,\beta\mp1}{\mathrm{ctg}\,\beta\pm\mathrm{ctg}\,\alpha}. \tag{1.15}$$

**Example 1.25.** Calculate $\sin(\alpha+\beta)$ if $\sin\alpha=\frac{3}{5}$; $\cos\beta=-\frac{5}{13}$; $\alpha\in\left(\frac{\pi}{2},\pi\right)$; $\beta\in\left(\pi,\frac{3\pi}{2}\right)$.

*Solution*. Let's find $\cos\alpha$ and $\sin\beta$ under conditions $\alpha\in\left(\frac{\pi}{2},\pi\right)$ and $\beta\in\left(\pi,\frac{3\pi}{2}\right)$.

Then $\cos\alpha=-\sqrt{1-\left(\frac{3}{5}\right)^2}=-\frac{4}{5}$, $\sin\beta=-\sqrt{1-\left(-\frac{5}{13}\right)^2}=-\frac{12}{13}$.

Using formula (1.9) we get:

$$\sin(\alpha+\beta)=\frac{3}{5}\cdot\left(-\frac{5}{13}\right)+\left(-\frac{4}{5}\right)\cdot\left(-\frac{12}{13}\right)=\frac{33}{65}.$$

**Example 1.26.** Calculate $\cos15^\circ$.

*Solution*. Let's write $\cos 15^\circ = \cos(60^\circ - 45^\circ)$. Using formula (1.12) we have:

$$\cos(60^\circ - 45^\circ) = \cos 60^\circ \cdot \cos 45^\circ + \sin 60^\circ \cdot \sin 45^\circ =$$
$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2} \cdot \left(\sqrt{3}+1\right)}{4}.$$

**Example 1.27.** Simplify $A = \dfrac{\sin\left(45^\circ + \alpha\right) - \cos\left(45^\circ + \alpha\right)}{\sin\left(45^\circ + \alpha\right) + \cos\left(45^\circ + \alpha\right)}$.

*Solution*.

$$A = \frac{\left(\sin 45^\circ \cdot \cos\alpha + \cos 45^\circ \cdot \sin\alpha\right) - \left(\cos 45^\circ \cdot \cos\alpha - \sin 45^\circ \cdot \sin\alpha\right)}{\left(\sin 45^\circ \cdot \cos\alpha + \cos 45^\circ \cdot \sin\alpha\right) + \left(\cos 45^\circ \cdot \cos\alpha - \sin 45^\circ \cdot \sin\alpha\right)} =$$
$$= \frac{\left(\frac{\sqrt{2}}{2} \cdot \cos\alpha + \frac{\sqrt{2}}{2} \cdot \sin\alpha\right) - \left(\frac{\sqrt{2}}{2} \cdot \cos\alpha - \frac{\sqrt{2}}{2} \cdot \sin\alpha\right)}{\left(\frac{\sqrt{2}}{2} \cdot \cos\alpha + \frac{\sqrt{2}}{2} \cdot \sin\alpha\right) + \left(\frac{\sqrt{2}}{2} \cdot \cos\alpha - \frac{\sqrt{2}}{2} \cdot \sin\alpha\right)} = \frac{\sqrt{2}\sin\alpha}{\sqrt{2}\cos\alpha} = \operatorname{tg}\alpha.$$

**Example 1.28.** Simplify $\frac{1}{2}\left(\cos\alpha + \sqrt{3}\sin\alpha\right)$.

*Solution.*

$$\frac{1}{2}\left(\cos\alpha + \sqrt{3}\sin\alpha\right) = \frac{1}{2}\cos\alpha + \frac{\sqrt{3}}{2}\sin\alpha =$$
$$= \cos 60^\circ \cdot \cos\alpha + \sin 60^\circ \cdot \sin\alpha = \cos\left(60^\circ - \alpha\right).$$

**Example 1.29.** Prove the identity:

$$\frac{\sqrt{2}\cos\alpha - 2\cos\left(45^\circ + \alpha\right)}{2\sin\left(45^\circ + \alpha\right) - \sqrt{2}\sin\alpha} = \operatorname{tg}\alpha.$$

*Solution*. Let's transform the left part of the equality:

$$\frac{\sqrt{2}\cos\alpha-2\cos\left(45^\circ+\alpha\right)}{2\sin\left(45^\circ+\alpha\right)-\sqrt{2}\sin\alpha}=\frac{\sqrt{2}\cos\alpha-2\left(\cos 45^\circ\cdot\cos\alpha-\sin 45^\circ\cdot\sin\alpha\right)}{2\left(\sin 45^\circ\cos\alpha-\cos 45^\circ\sin\alpha\right)-\sqrt{2}\sin\alpha}=$$

$$=\frac{\sqrt{2}\cos\alpha-2\frac{\sqrt{2}}{2}\cdot\cos\alpha+2\frac{\sqrt{2}}{2}\cdot\sin\alpha}{2\left(\frac{\sqrt{2}}{2}\cos\alpha-\frac{\sqrt{2}}{2}\sin\alpha\right)-\sqrt{2}\sin\alpha}=\frac{\sqrt{2}\sin\alpha}{\sqrt{2}\cos\alpha}=\operatorname{tg}\alpha,$$

which was to be proved.

## Tasks for individual work

**Task 1.30.** Calculate the most rational way:

1) $\cos 75^\circ\cdot\cos 15-\sin 75^\circ\cdot\sin 15^\circ$;

2) $\sin 22^\circ+\sin 50^\circ\cos 28^\circ-\cos 50^\circ\cdot\sin 28^\circ$;

3) $\sin(10^\circ+\alpha)\cos(20^\circ-\alpha)+\cos(10^\circ+\alpha)\cdot\sin(20^\circ-\alpha)$.

**Task 1.31.** Calculate:

1) $\sin(\alpha-\beta)$ if $\cos\alpha=-\frac{4}{5}$, $\sin\beta=-\frac{24}{25}$, $\alpha\in\left(\pi,\frac{3\pi}{2}\right)$; $\beta\in\left(\frac{3\pi}{2},2\pi\right)$;

2) $\cos(\alpha+\beta)$ if $\operatorname{tg}\alpha=-\frac{24}{7}$, $\operatorname{tg}\beta=\frac{15}{8}$, $\alpha\in\left(\frac{\pi}{2},\pi\right)$; $\beta\in\left(\pi,\frac{3\pi}{2}\right)$;

3) $\sin 105^\circ$; 4) $\operatorname{tg}15^\circ$.

**Task 1.32.** Simplify:

1) $\sin\alpha\cdot\cos 3\alpha-\cos\alpha\cdot\sin 3\alpha$; 2) $\cos\left(\alpha+\frac{\pi}{6}\right)+\cos\left(\alpha-\frac{\pi}{6}\right)$;

3) $\cos(\alpha+\beta)+\cos(\alpha-\beta)$; 4) $\frac{\cos\alpha\cos\beta-\cos(\alpha+\beta)}{\cos(\alpha-\beta)-\sin\alpha\cdot\sin\beta}$;

5) $\frac{2\sin\alpha\cdot\cos\beta-\sin(\alpha-\beta)}{\cos(\alpha-\beta)-2\sin\alpha\cdot\sin\beta}$.

**Task 1.33.** Simplify:

1) $\sin 15^\circ+\operatorname{tg}30^\circ\cos 15^\circ$; 2) $\sin 3\alpha+\operatorname{tg}\frac{\pi}{4}\cos 3\alpha$;

3) $\cos 15^\circ + \sqrt{3}\sin 15^\circ$; 4) $\dfrac{\sqrt{3} - \operatorname{tg} 20^\circ}{1 + \sqrt{3} \cdot \operatorname{tg} 20^\circ}$; 5) $\sin\alpha + \sqrt{3}\cos\alpha$.

**Task 1.34.** Prove the identities:

1) $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta$;

2) $\cos(\alpha+45^\circ) - \cos(\alpha-45^\circ) = -\sqrt{2}\cdot\sin\alpha$;

3) $0.5\cdot\cos\alpha + \dfrac{\sqrt{3}}{2}\sin\alpha = \sin(30^\circ+\alpha)$;

4) $\dfrac{\sin(\alpha-\beta)}{\cos\alpha\cdot\cos\beta} = \operatorname{tg}\alpha - \operatorname{tg}\beta$;

5) $\sin\alpha\cdot\left(1 + \operatorname{tg}\alpha\cdot\operatorname{tg}\dfrac{\alpha}{2}\right) = \operatorname{tg}\alpha$;

6) $\dfrac{\operatorname{tg}\alpha + \operatorname{tg}\beta}{\operatorname{tg}\alpha - \operatorname{tg}\beta} = \dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$.

## 1.7. Reduction formulas

Reduction formulas make it possible to express trigonometric functions of angles of the kind $\left(\dfrac{\pi}{2}\pm\alpha\right)$, $(\pi+\alpha)$, $\left(\dfrac{3\pi}{2}\pm\alpha\right)$, $(2\pi\pm\alpha)$ through trigonometric functions of the angle $\alpha$ using the reductions rules:

a) the trigonometric functions of the angles of the kind $\left(\dfrac{\pi}{2}\pm\alpha\right)$, $\left(\dfrac{3\pi}{2}\pm\alpha\right)$ are transformed into the trigonometric functions of the angle $\alpha$, but of the opposite name (sine into cosine, cosine into sine, tangent into cotangent, cotangent into tangent);

b) the trigonometric functions of the angles of the kind $(\pi+\alpha)$, $(2\pi\pm\alpha)$ are transformed into the trigonometric functions of the angle $\alpha$ of the same name;

c) the sign of the obtained trigonometric function of the angle $\alpha$ is defined by the sign of the initial function if the angle $\alpha$ is conditionally acute.

It is usefull to use the formulas of an additional angle:

$$\sin\alpha = \cos\left(\frac{\pi}{2} - \alpha\right); \qquad \cos\alpha = \sin\left(\frac{\pi}{2} - \alpha\right);$$

$$\mathrm{tg}\,\alpha = \mathrm{ctg}\left(\frac{\pi}{2} - \alpha\right); \qquad \mathrm{ctg}\,\alpha = \mathrm{tg}\left(\frac{\pi}{2} - \alpha\right).$$

For example,

$$\sin 60^\circ = \cos\left(90^\circ - 60^\circ\right) = \cos 30^\circ; \quad \mathrm{tg}\left(73^\circ\right) = \mathrm{ctg}\left(90^\circ - 73^\circ\right) = \mathrm{ctg}\,17^\circ.$$

All the basic reduction formulas are given in Table 1.2.

Table 1.2.

**Reduction formulas for trigonometric functions**

| Angle $x$ / Function | $(90^\circ - \alpha)$ | $(90^\circ + \alpha)$ | $(180^\circ - \alpha)$ | $(180^\circ + \alpha)$ | $(270^\circ - \alpha)$ | $(270^\circ + \alpha)$ | $(360^\circ - \alpha)$ | $(360^\circ + \alpha)$ |
|---|---|---|---|---|---|---|---|---|
| | $\frac{\pi}{2} - \alpha$ | $\frac{\pi}{2} + \alpha$ | $\pi - \alpha$ | $\pi + \alpha$ | $\frac{3\pi}{2} - \alpha$ | $\frac{3\pi}{2} + \alpha$ | $2\pi - \alpha$ | $2\pi + \alpha$ |
| $\sin x$ | $\cos\alpha$ | $\cos\alpha$ | $\sin\alpha$ | $-\sin\alpha$ | $-\cos\alpha$ | $-\cos\alpha$ | $-\sin\alpha$ | $\sin\alpha$ |
| $\cos x$ | $\sin\alpha$ | $-\sin\alpha$ | $-\cos\alpha$ | $-\cos\alpha$ | $-\sin\alpha$ | $\sin\alpha$ | $\cos\alpha$ | $\cos\alpha$ |
| $\mathrm{tg}\,x$ | $\mathrm{ctg}\,\alpha$ | $-\mathrm{ctg}\,\alpha$ | $-\mathrm{tg}\,\alpha$ | $\mathrm{tg}\,\alpha$ | $\mathrm{ctg}\,\alpha$ | $-\mathrm{ctg}\,\alpha$ | $-\mathrm{tg}\,\alpha$ | $\mathrm{tg}\,\alpha$ |
| $\mathrm{ctg}\,x$ | $\mathrm{tg}\,\alpha$ | $-\mathrm{tg}\,\alpha$ | $-\mathrm{ctg}\,\alpha$ | $\mathrm{ctg}\,\alpha$ | $\mathrm{tg}\,\alpha$ | $-\mathrm{tg}\,\alpha$ | $-\mathrm{ctg}\,\alpha$ | $\mathrm{ctg}\,\alpha$ |

**Example 1.35.** Calculate $\sin 225^\circ$.

*Solution*. We have $\sin 225^\circ = \sin\left(180^\circ + 45^\circ\right) = -\sin 45^\circ = -\frac{\sqrt{2}}{2}$.

**Example 1.36.** Calculate:

$$\sin 390^\circ \cdot \sin 150^\circ + \cos 210^\circ \cdot \cos 150^\circ + \mathrm{tg}\,240^\circ \cdot \mathrm{tg}\,210^\circ.$$

*Solution*. We have

$$\sin\left(360^{o}+30^{o}\right)\cdot\sin(180^{o}-30^{o})+\cos(180^{o}+30^{o})\cdot\cos(180^{o}-30^{o})+$$
$$+\operatorname{tg}\left(270^{o}-30^{o}\right)\cdot\operatorname{tg}\left(180^{o}+30^{o}\right)=\sin 30^{o}\cdot\sin 30^{o}+\left(-\cos 30^{o}\right)\cdot\left(-\cos 30^{o}\right)+$$
$$+\operatorname{ctg}30^{o}\cdot\operatorname{tg}30^{o}=\sin^{2}30^{o}+\cos^{2}30^{o}+1=1+1=2.$$

**Example 1.37.** Simplify the expression:

$$A=\frac{\operatorname{tg}\left(180^{\circ}-\alpha\right)\cdot\cos\left(180^{\circ}-\alpha\right)\cdot\operatorname{tg}\left(90^{\circ}-\alpha\right)}{\sin\left(90^{\circ}+\alpha\right)\cdot\operatorname{ctg}\left(90^{\circ}+\alpha\right)\cdot\operatorname{tg}\left(90^{\circ}+\alpha\right)}.$$

*Solution*. $A=\dfrac{(-\operatorname{tg}\alpha)\cdot(-\cos\alpha)\cdot\operatorname{ctg}\alpha}{\cos\alpha\cdot(-\operatorname{tg}\alpha)\cdot(-\operatorname{ctg}\alpha)}=1.$

**Example 1.38.** Simplify:

$$A=\frac{\sin 515^{\circ}\cdot\cos\left(-475^{\circ}\right)+\operatorname{ctg}222^{\circ}\cdot\operatorname{ctg}408^{\circ}}{\operatorname{ctg}415^{\circ}\operatorname{ctg}\left(-505^{\circ}\right)+\operatorname{tg}197^{\circ}\cdot\operatorname{tg}73^{\circ}}.$$

*Solution*.

$$A=\frac{\sin\left(360^{\circ}+155^{\circ}\right)\cdot\cos\left(360^{\circ}+115^{\circ}\right)+\operatorname{ctg}\left(180^{\circ}+42^{\circ}\right)\cdot\operatorname{ctg}\left(360^{\circ}+48^{\circ}\right)}{\operatorname{ctg}\left(360^{\circ}+55^{\circ}\right)\left(-\operatorname{ctg}\left(360^{\circ}+145^{\circ}\right)\right)+\operatorname{tg}\left(180^{\circ}+17^{\circ}\right)\cdot\operatorname{ctg}\left(90^{\circ}-73^{\circ}\right)}=$$

$$=\frac{\sin\left(155^{\circ}\right)\cdot\cos\left(115^{\circ}\right)+\operatorname{ctg}\left(42^{\circ}\right)\cdot\operatorname{ctg}\left(48^{\circ}\right)}{\operatorname{ctg}\left(55^{\circ}\right)\left(-\operatorname{ctg}\left(145^{\circ}\right)\right)+\operatorname{tg}\left(17^{\circ}\right)\cdot\operatorname{ctg}\left(17^{\circ}\right)}=$$

$$=\frac{\sin\left(180^{o}-25^{o}\right)\cdot\cos\left(90^{o}+25^{o}\right)+\operatorname{ctg}42^{o}\cdot\operatorname{tg}42^{o}}{\operatorname{ctg}55^{o}\left(-\operatorname{ctg}\left(90^{o}+55^{o}\right)\right)+1}=$$

$$=\frac{\sin 25^{o}\cdot\left(-\sin 25^{o}\right)+1}{-\operatorname{ctg}55^{o}\left(-\operatorname{tg}55^{o}\right)+1}=\frac{1-\sin^{2}25^{o}}{2}=\frac{\cos^{2}25^{o}}{2}.$$

**Example 1.39.** Prove the identity:

$$\frac{\cos(5\pi-\alpha)\cdot\sin\left(\frac{\pi}{2}+\alpha\right)}{\mathrm{tg}\left(\frac{3\pi}{2}+\alpha\right)\mathrm{tg}(2\pi+\alpha)}-\cos^2\left(\frac{3\pi}{2}-\alpha\right)=\cos^2\alpha-\sin^2\alpha.$$

*Solution*. Let's transform the left part of the equality:

$$\frac{\cos(5\pi-\alpha)\cdot\sin\left(\frac{\pi}{2}+\alpha\right)}{\mathrm{tg}\left(\frac{3\pi}{2}+\alpha\right)\mathrm{tg}(2\pi+\alpha)}-\cos^2\left(\frac{3\pi}{2}-\alpha\right)=\frac{-\cos\alpha\cdot\cos\alpha}{-\mathrm{ctg}\,\alpha\cdot\mathrm{tg}\,\alpha}-\sin^2\alpha=$$

$$=\cos^2\alpha-\sin^2\alpha,$$

which was to be proved.

## Tasks for individual work

**Task 1.40.** Calculate the values using the reduction formulas.

1) $\sin 135^\circ$; 2) $\mathrm{ctg}\,150^\circ$; 3) $\cos 240^\circ$; 4) $\mathrm{tg}\,320^\circ$;

5) $\sin^2 99^\circ+\cos^2 81^\circ+\mathrm{ctg}^2\,315^\circ$;

6) $\mathrm{tg}\,18^\circ\cdot\mathrm{tg}\,288^\circ+\sin 32^\circ\cdot\sin 148^\circ-\sin 302^\circ\cdot\sin 122^\circ$;

7) $\sin 575^\circ\cdot\cos 845^\circ+\sin 1405^\circ\cdot\sin 1675^\circ-\mathrm{tg}\,215^\circ\cdot\mathrm{tg}\,685^\circ-\mathrm{tg}^2\,35^\circ$.

**Task 1.41.** Simplify the expressions:

1) $\sin(180^\circ-\alpha)+\cos(90^\circ+\alpha)-\mathrm{tg}(360^\circ-\alpha)+\mathrm{ctg}(270^\circ-\alpha)$;

2) $\sin^2(180^\circ-\alpha)+\sin^2(270^\circ-\alpha)+\mathrm{tg}(90^\circ+\alpha)\cdot\mathrm{ctg}(360^\circ-\alpha)$;

3) $\sin(5\alpha-\pi)\cdot\cos(\alpha-3\pi)+\sin(5\alpha-5{,}5\pi)\cdot\cos(\alpha+0{,}5\pi)$;

4) $\dfrac{\mathrm{tg}\left(\alpha-\frac{\pi}{2}\right)-\mathrm{ctg}(\pi-\alpha)+\cos\left(\alpha-\frac{3\pi}{2}\right)}{\sin(\pi+\alpha)}$;

5) $\dfrac{\sin\left(\frac{3\pi}{2}-\alpha\right)\cdot\cos\left(\frac{\pi}{2}+\alpha\right)}{\mathrm{tg}(\pi-\alpha)}$;

6) $\dfrac{\sin(3\pi+\alpha)\cdot\cos\left(\dfrac{3\pi}{2}-\alpha\right)}{\operatorname{tg}(\pi-2\alpha)}$.

**Task 1.42.** Prove the identities:

1) $\dfrac{\sin(\alpha-2\pi)-\sin\left(\dfrac{\pi}{2}-\alpha\right)}{\cos\left(\dfrac{\pi}{2}-\alpha\right)+\sin\alpha}=\dfrac{1}{2}(1-\operatorname{ctg}\alpha);$

2) $\dfrac{\sin(\pi+\alpha)}{\operatorname{ctg}\left(\dfrac{\pi}{2}-\alpha\right)}\cdot\dfrac{\operatorname{tg}(\alpha-\pi)}{\operatorname{ctg}(\pi+\alpha)}\cdot\dfrac{\cos(2\pi-\alpha)}{\cos\left(\dfrac{3\pi}{2}-\alpha\right)}=\sin\alpha.$

## 1.8. Trigonometric functions of multiple arguments

1. Formulas of double arguments. If we consider formulas (1.9), (1.11), (1.13) for sine, cosine and tangent of a sum of two angles and take $\alpha=\beta$, we get formulas of double arguments:

$$\sin 2\alpha = 2\sin\alpha\cdot\cos\alpha; \tag{1.16}$$

$$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha=1-2\sin^2\alpha=2\cos^2\alpha-1; \tag{1.17}$$

$$\operatorname{tg}2\alpha=\frac{2\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha}. \tag{1.18}$$

2. Formulas of half arguments. Using formulas (1.17) we get two basic formulas:

$$\sin^2\alpha=\frac{1-\cos 2\alpha}{2};\quad \cos^2\alpha=\frac{1+\cos 2\alpha}{2}; \tag{1.19}$$

$$1-\cos 2\alpha=2\sin^2\alpha;\quad 1+\cos 2\alpha=2\cos^2\alpha. \tag{1.20}$$

If the angle of the left side of the formula equals a half of the angle of the right part of the formula, then these formulas are called formulas of half arguments. These are formulas (1.19), we can rewrite them in the form:

$$\sin^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{2}; \tag{1.21}$$

$$\cos^2\frac{\alpha}{2}=\frac{1+\cos\alpha}{2}; \tag{1.22}$$

$$\operatorname{tg}^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{1+\cos\alpha}. \tag{1.23}$$

It is easy to show that

$$\operatorname{tg}\frac{\alpha}{2}=\frac{1-\cos\alpha}{\sin\alpha}.$$

3. Formulas of triple arguments. One should know the basic formulas of triple arguments:

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha; \tag{1.24}$$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha. \tag{1.25}$$

4. Expression of trigonometric functions through the tangent of a half angle using the formulas:

$$\sin\alpha=\frac{2\operatorname{tg}\frac{\alpha}{2}}{1+\operatorname{tg}^2\frac{\alpha}{2}};\qquad \cos\alpha=\frac{1-\operatorname{tg}^2\frac{\alpha}{2}}{1+\operatorname{tg}^2\frac{\alpha}{2}}; \tag{1.26}$$

$$\operatorname{tg}\alpha=\frac{1-\operatorname{tg}^2\frac{\alpha}{2}}{2\operatorname{tg}\frac{\alpha}{2}}. \tag{1.27}$$

**Example 1.43.** Calculate the values without tables:

$$\sin 75^\circ\cdot\sin 15^\circ.$$

*Solution*.

$$\sin 75^\circ \cdot \sin 15^\circ = \sin\left(90^\circ - 15^\circ\right) \cdot \sin 15^\circ = \cos 15^\circ \cdot \sin 15^\circ =$$

$$= \frac{2\cos 15^\circ \cdot \sin 15^\circ}{2} = \frac{\sin 30^\circ}{2} = \frac{1}{4}.$$

**Example 1.44.** Calculate the values without tables:

$$4\sin 18^\circ \cdot \sin 306^\circ.$$

*Solution*.

$$4\sin 18^\circ \cdot \sin 306^\circ = 4\sin 18^\circ \cdot \sin\left(270^\circ + 36^\circ\right) = -4\sin 18^\circ \cdot \cos 36^\circ =$$

$$= \frac{-4\sin 18^\circ \cdot \cos 18^\circ \cdot \cos 36^\circ}{\cos 18^\circ} = \frac{-2\sin 36^\circ \cdot \cos 36^\circ}{\cos 18^\circ} =$$

$$= -\frac{\sin 72^\circ}{\cos 18^\circ} = -\frac{\cos 18^\circ}{\cos 18^\circ} = -1.$$

**Example 1.45.** Simplify:

$$\frac{1 + \sin 2\alpha - \cos 2\alpha}{1 + \sin 2\alpha + \cos 2\alpha}.$$

*Solution*. Let's transform the fraction:

$$\frac{1 + \sin 2\alpha - \cos 2\alpha}{1 + \sin 2\alpha + \cos 2\alpha} = \frac{\left(1 - \cos 2\alpha\right) + \sin 2\alpha}{\left(1 + \cos 2\alpha\right) + \sin 2\alpha} = \frac{2\sin^2 \alpha + 2\sin \alpha \cos \alpha}{2\cos^2 \alpha + 2\sin \alpha \cos \alpha} =$$

$$= \frac{2\sin \alpha\left(\sin \alpha + \cos \alpha\right)}{2\cos \alpha\left(\sin \alpha + \cos \alpha\right)} = \frac{\sin \alpha}{\cos \alpha} = \operatorname{tg} \alpha.$$

**Example 1.46.** Calculate $\dfrac{5\cos \alpha - 3}{10\sin \alpha + 1}$ if $\operatorname{tg}\dfrac{\alpha}{2} = 3$.

*Solution*. According to the formula (1.26) we have:

$$\cos \alpha = \frac{1 - \operatorname{tg}^2 \frac{\alpha}{2}}{1 + \operatorname{tg}^2 \frac{\alpha}{2}} = \frac{1 - 9}{1 + 9} = -\frac{4}{5}; \qquad \sin \alpha = \frac{2\operatorname{tg} \frac{\alpha}{2}}{1 + \operatorname{tg}^2 \frac{\alpha}{2}} = \frac{6}{1 + 9} = \frac{3}{5}.$$

Then $$\frac{5\cos\alpha-3}{10\sin\alpha+1}=\frac{5\cdot\left(-\frac{4}{5}\right)-3}{10\cdot\frac{3}{5}+1}=\frac{-7}{7}=-1.$$

**Example 1.47.** Simplify $1-\cos\left(\frac{\alpha}{2}-3\pi\right)-\cos^2\frac{\alpha}{4}+\sin^2\frac{\alpha}{4}$.

*Solution*. We have: $1-\cos\left(\frac{\alpha}{2}-3\pi\right)-\cos^2\frac{\alpha}{4}+\sin^2\frac{\alpha}{4}=$

$$=1-\cos\left(\frac{\alpha}{2}-3\pi\right)-\left(\cos^2\frac{\alpha}{4}-\sin^2\frac{\alpha}{4}\right)=1+\cos\frac{\alpha}{2}-\cos\frac{\alpha}{2}=1.$$

**Example 1.48.** Calculate the value $\operatorname{tg}112^\circ30'$ without tables.

*Solution*. According to formula (1.23) $\operatorname{tg}\frac{\alpha}{2}=\frac{1-\cos\alpha}{\sin\alpha}$ we have:

$$\operatorname{tg}112^\circ30'=\frac{1-\cos225^\circ}{\sin225^\circ}=\frac{1-\cos(180^\circ+45^\circ)}{\sin(180^\circ+45^\circ)}=\frac{1+\cos45^\circ}{-\sin45^\circ}=$$

$$=\frac{1+\sqrt{2}/2}{-\sqrt{2}/2}=\frac{2+\sqrt{2}}{-\sqrt{2}}=-\left(1+\sqrt{2}\right).$$

**Example 1.49.** Prove the identity:

$$\frac{\sin^4\alpha+2\sin\alpha\cos\alpha-\cos^4\alpha}{\operatorname{tg}2\alpha-1}=\cos2\alpha.$$

*Solution*. Let's transform the left part of this identity:

$$\frac{\sin^4\alpha+2\sin\alpha\cos\alpha-\cos^4\alpha}{\operatorname{tg}2\alpha-1}=\frac{2\sin\alpha\cos\alpha-\left(\cos^4\alpha-\sin^4\alpha\right)}{\frac{\sin2\alpha}{\cos2\alpha}-1}=$$

$$= \frac{2\sin\alpha\cos\alpha - \left(\cos^2\alpha - \sin^2\alpha\right)\left(\cos^2\alpha + \sin^2\alpha\right)}{\dfrac{\sin 2\alpha - \cos 2\alpha}{\cos 2\alpha}} =$$

$$= \frac{\sin 2\alpha - \cos 2\alpha}{\sin 2\alpha - \cos 2\alpha}\cos 2\alpha = \cos 2\alpha,$$

which was to be proved.

**Example 1.50.** Simplify the expression:

$$3\left(\sin^4\alpha + \cos^4\alpha\right) - 2\left(\sin^6\alpha + \cos^6\alpha\right).$$

*Solution*. We have:

$$3\left(\sin^4\alpha + \cos^4\alpha\right) - 2\left(\sin^6\alpha + \cos^6\alpha\right) = 3\left(\sin^4\alpha + \cos^4\alpha\right) -$$

$$-2\left(\left(\sin^2\alpha\right)^3 + \left(\cos^2\alpha\right)^3\right) = 3\left(\sin^4\alpha + \cos^4\alpha\right) - 2\left(\sin^2\alpha + \cos^2\alpha\right)\cdot$$

$$\cdot\left(\sin^4\alpha - \sin^2\alpha\cos^2\alpha + \cos^4\alpha\right) = 3\left(\sin^4\alpha + \cos^4\alpha\right) - 2\left(\sin^4\alpha + \cos^4\alpha\right) +$$

$$+ 2\sin^2\alpha\cos^2\alpha = \left(\sin^2\alpha + \cos^2\alpha\right)^2 = 1.$$

**Example 1.51.** Simplify:

$$A = \frac{1 + \cos(4\alpha - 2\pi) + \cos\left(4\alpha - \dfrac{\pi}{2}\right)}{1 + \cos(4\alpha + 3\pi) + \cos\left(4\alpha + \dfrac{3\pi}{2}\right)}.$$

*Solution*.

$$A = \frac{1 + \cos(4\alpha - 2\pi) + \cos\left(4\alpha - \dfrac{\pi}{2}\right)}{1 + \cos(4\alpha + 3\pi) + \cos\left(4\alpha + \dfrac{3\pi}{2}\right)} = \frac{1 + \cos 4\alpha + \sin 4\alpha}{1 - \cos 4\alpha + \sin 4\alpha} =$$

$$= \frac{2\cos^2 2\alpha + 2\sin 2\alpha\cos 2\alpha}{2\sin^2 2\alpha + 2\sin 2\alpha\cos 2\alpha} = \frac{2\cos 2\alpha\left(\cos 2\alpha + \sin 2\alpha\right)}{2\sin 2\alpha\left(\cos 2\alpha + \sin 2\alpha\right)} = \operatorname{ctg} 2\alpha.$$

**Example 1.52.** Prove the identity:

$$\frac{1-2\cos^2\alpha}{2\,\mathrm{tg}\left(\alpha-\frac{\pi}{4}\right)\sin^2\left(\frac{\pi}{4}+\alpha\right)}=1.$$

*Solution*. Let's transform the left part of the identity:

$$\frac{1-2\cos^2 2\alpha}{2\,\mathrm{tg}\left(\alpha-\frac{\pi}{4}\right)\sin^2\left(\frac{\pi}{4}+\alpha\right)}=\frac{1-2\cos^2 2\alpha}{2\,\mathrm{tg}\left(\alpha-\frac{\pi}{4}\right)\cos^2\left(\frac{\pi}{2}-\left(\frac{\pi}{4}-\alpha\right)\right)}=$$

$$=\frac{1-2\cos^2 2\alpha}{2\frac{\sin\left(\alpha-\frac{\pi}{4}\right)}{\cos\left(\alpha-\frac{\pi}{4}\right)}\cos^2\left(\frac{\pi}{2}-\left(\frac{\pi}{4}-\alpha\right)\right)}=\frac{1-2\cos^2 2\alpha}{2\sin\left(\alpha-\frac{\pi}{4}\right)\cos\left(\alpha-\frac{\pi}{4}\right)}=$$

$$=\frac{1-2\cos^2 2\alpha}{\sin\left(2\alpha-\frac{\pi}{2}\right)}=\frac{1-(1+\cos 2\alpha)}{-\sin\left(\frac{\pi}{2}-2\alpha\right)}=\frac{-\cos 2\alpha}{-\cos 2\alpha}=1,$$

which was to be proved.

## Tasks for individual work

**Task 1.53.** Simplify the expressions:

1) $\dfrac{1+\sin 2\alpha}{(\sin\alpha+\cos\alpha)^2}$; 2) $\dfrac{4\sin^4\alpha+\sin^2 2\alpha}{\sin\alpha}$;

3) $\dfrac{\cos 4\alpha+1}{\mathrm{ctg}\,\alpha-\mathrm{tg}\,\alpha}$; 4) $\dfrac{\sin 2\alpha}{\sin\alpha}-\dfrac{\cos 2\alpha}{\cos\alpha}$;

5) $2\sin 2\alpha\cdot\cos 2\alpha\cdot\left(\cos^2 2\alpha-\sin^2 2\alpha\right)$;

6) $2\cos^2\left(\frac{\pi}{4}-\alpha\right)-\sin 2\alpha$; 7) $\dfrac{\sin 2\alpha-2\sin\alpha}{\cos\alpha-1}$;

8) $\left(\dfrac{\sin\alpha}{1+\cos\alpha}+\dfrac{1+\cos\alpha}{\sin\alpha}\right)\cdot\sin 2\alpha$;

9) $\left(\dfrac{4\sin\alpha\cdot\cos\alpha\cdot\cos 2\alpha}{\sin^2 2\alpha-\cos^2 2\alpha}\right)\cdot\operatorname{ctg}4\alpha;$

10) $\dfrac{\sin 6\alpha}{\sin 2\alpha}+\dfrac{\cos(6\alpha-\pi)}{\cos 2\alpha};$

11) $\left(\dfrac{\cos^2\alpha-\cos^4\alpha}{\sin^2\alpha-\sin^4\alpha}-\cos^2\alpha\right)\cdot 4\cos^2\alpha;$

12) $\dfrac{(\cos 2\alpha+\sin 2\alpha)^2-(\cos\alpha+\sin\alpha)^2+\sin 2\alpha}{(\cos 2\alpha+\sin 2\alpha)\cdot(\cos 2\alpha-\sin 2\alpha)};$

13) $\dfrac{\cos^3\alpha-\cos 3\alpha}{\sin^3\alpha+\sin 3\alpha}.$

**Task 1.54.** Calculate the values without tables:

1) $\sin 2\alpha,\ \ \cos 2\alpha,\ \ \operatorname{tg}2\alpha$ if $\sin\alpha=\dfrac{4}{5}$ and $\alpha\in\left(0;\dfrac{\pi}{2}\right);$

2) $\sin\dfrac{\alpha}{2},\ \ \cos\dfrac{\alpha}{2},\ \ \operatorname{tg}\dfrac{\alpha}{2}$ if $\cos\alpha=\dfrac{1}{52}$ and $\alpha\in\left(0;\dfrac{\pi}{2}\right);$

3) $\dfrac{5\cos\alpha+4}{10\sin\alpha-1}$ if $\operatorname{tg}\alpha=2;$

4) $(\cos 15^\circ+\sin 15^\circ)^2;$ 5) $(\cos 75^\circ-\sin 75^\circ)^2;$

6) $\dfrac{2\operatorname{tg}15^\circ}{1-\operatorname{tg}^2 15^\circ};$ 7) $\cos\dfrac{3\pi}{5}\cdot\sin\dfrac{6\pi}{5};$ 8) $(\operatorname{tg}255^\circ-\operatorname{tg}555^\circ)(\operatorname{tg}795^\circ-\operatorname{tg}195^\circ);$

9) $\sqrt{\dfrac{2\sin\alpha-\sin 2\alpha}{2\sin\alpha+\sin 2\alpha}}$ if $\alpha\in(0;\pi).$

**Task 1.55.** Prove the following identities:

1) $\dfrac{2\sin\alpha-\sin 2\alpha}{2\sin\alpha+\sin 2\alpha}=\operatorname{tg}^2\dfrac{\alpha}{2};$ 2) $\dfrac{\sin 2\alpha}{1+\cos 2\alpha}:\dfrac{1+\cos\alpha}{\cos\alpha}=\operatorname{tg}\dfrac{\alpha}{2};$

3) $\dfrac{\cos\alpha-\cos 2\alpha-1}{\sin\alpha-\sin 2\alpha}=\operatorname{ctg}\alpha;$ 4) $\dfrac{\sin^3\alpha+\sin 3\alpha}{\sin\alpha}+\dfrac{\cos^3\alpha-\cos 3\alpha}{\cos\alpha}=3;$

5) $\cdot\sin^6 2\alpha+\cos^6 2\alpha+3\sin^2\alpha\cdot\cos^2\alpha=1;$

6) $\dfrac{1-\cos(8\alpha-3\pi)}{\operatorname{tg}2\alpha-\operatorname{ctg}2\alpha}=-\dfrac{1}{2}\sin 8\alpha.$

## 1.9. Transformation formulas

The formulas for transformation of a sum of trigonometric functions to a product and formulas for transformation of a product to a sum.

1. For transformation of an algebraic sum of trigonometric functions to a product the following formulas are used:

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2} \qquad (1.28)$$

$$\sin\alpha-\sin\beta=2\sin\frac{\alpha-\beta}{2}\cdot\cos\frac{\alpha+\beta}{2} \qquad (1.29)$$

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2} \qquad (1.30)$$

$$\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2} \qquad (1.31)$$

$$\sin\alpha+\cos\alpha=\sqrt{2}\sin\left(\alpha+\frac{\pi}{4}\right) \qquad (1.32)$$

$$\sin\alpha-\cos\alpha=\sqrt{2}\sin\left(\alpha-\frac{\pi}{4}\right) \qquad (1.33)$$

$$\operatorname{tg}\alpha+\operatorname{tg}\beta=\frac{\sin(\alpha+\beta)}{\cos\alpha\cdot\cos\beta}; \qquad (1.34)$$

$$\operatorname{tg}\alpha-\operatorname{tg}\beta=\frac{\sin(\alpha-\beta)}{\cos\alpha\cdot\cos\beta}; \qquad (1.35)$$

**Example 1.56.** Simplify the expression:

$$\frac{\sin 2\alpha+\sin 4\alpha+\sin 6\alpha}{\cos 2\alpha+\cos 4\alpha+\cos 6\alpha}.$$

*Solution*.

$$\frac{\sin 2\alpha+\sin 4\alpha+\sin 6\alpha}{\cos 2\alpha+\cos 4\alpha+\cos 6\alpha}=\frac{(\sin 6\alpha+\sin 2\alpha)+\sin 4\alpha}{(\cos 6\alpha+\cos 2\alpha)+\cos 4\alpha}=$$

$$=\frac{(2\sin 4\alpha\cos 2\alpha)+\sin 4\alpha}{(2\cos 4\alpha\cos 2\alpha)+\cos 4\alpha}=\frac{\sin 4\alpha(2\cos 2\alpha+1)}{\cos 4\alpha(2\cos 2\alpha+1)}=\operatorname{tg} 4\alpha.$$

**Example 1.57.** Transform the expression to a product:

$$\operatorname{tg}\alpha+\operatorname{tg}2\alpha-\operatorname{tg}3\alpha.$$

*Solution*. $(\operatorname{tg}\alpha+\operatorname{tg}2\alpha)-\operatorname{tg}3\alpha=\dfrac{\sin 3\alpha}{\cos\alpha\cdot\cos 2\alpha}-\dfrac{\sin 3\alpha}{\cos 3\alpha}=$

$$=\frac{\sin 3\alpha\cdot(\cos 3\alpha-\cos\alpha\cdot\cos 2\alpha)}{\cos\alpha\cdot\cos 2\alpha\cdot\cos 3\alpha}=$$

$$=\frac{\sin 3\alpha\cdot(\cos(\alpha+2\alpha)-\cos\alpha\cdot\cos 2\alpha)}{\cos\alpha\cdot\cos 2\alpha\cdot\cos 3\alpha}=$$

$$=\frac{\sin 3\alpha\cdot(\cos\alpha\cdot\cos 2\alpha-\sin\alpha\cdot\sin 2\alpha-\cos\alpha\cdot\cos 2\alpha)}{\cos\alpha\cdot\cos 2\alpha\cdot\cos 3\alpha}=$$

$$=-\frac{\sin 3\alpha\cdot(\sin\alpha\cdot\sin 2\alpha)}{\cos\alpha\cdot\cos 2\alpha\cdot\cos 3\alpha}=-\operatorname{tg}\alpha\cdot\operatorname{tg}2\alpha\cdot\operatorname{tg}3\alpha.$$

**Example 1.58.** Transform the expression to a product:

$$\sin^2\alpha+\sin^2 2\alpha-\sin^2 3\alpha-\sin^2 4\alpha.$$

*Solution*. Using the formula of reduction of the power $\sin^2\dfrac{\alpha}{2}=\dfrac{1-\cos\alpha}{2}$ we get:

$$\sin^2\alpha+\sin^2 2\alpha-\sin^2 3\alpha-\sin^2 4\alpha=\frac{1-\cos 2\alpha}{2}+\frac{1-\cos 4\alpha}{2}-\frac{1-\cos 6\alpha}{2}-$$

$$-\frac{1-\cos 8\alpha}{2}=\frac{1}{2}(\cos 8\alpha+\cos 6\alpha)-\frac{1}{2}(\cos 4\alpha+\cos 2\alpha)=\cos 7\alpha\cdot\cos\alpha-$$

$$-\cos 3\alpha\cdot\cos\alpha=\cos\alpha\cdot(\cos 7\alpha-\cos 3\alpha)=-2\cos\alpha\cdot\sin 5\alpha\cdot\sin 2\alpha.$$

**Example 1.59.** Transform the expressions to a product:

a) $1+\sin 2\alpha+\cos 2\alpha;$ b) $\sqrt{3}-2\sin\alpha.$

*Solution*.

$$1+\sin 2\alpha+\cos 2\alpha=(1+\cos 2\alpha)+\sin 2\alpha=2\cos^2\alpha+$$

a) $$+2\sin\alpha\cdot\cos\alpha=2\cos\alpha\cdot(\sin\alpha+\cos\alpha)=2\cos\alpha\cdot\sqrt{2}\sin\left(\alpha+\frac{\pi}{4}\right)=$$

$$=2\sqrt{2}\cos\alpha\cdot\sin\left(\alpha+\frac{\pi}{4}\right).$$

b) $$\sqrt{3}-2\sin\alpha=2\left(\frac{\sqrt{3}}{2}-\sin\alpha\right)=2(\sin 60^\circ-\sin\alpha)=$$

$$=4\sin\left(30^\circ-\frac{\alpha}{2}\right)\cos\left(30^\circ+\frac{\alpha}{2}\right).$$

Transform the product of the trigonometric functions to a sum using the following formulas:

$$\sin\alpha\cdot\cos\beta=\frac{1}{2}(\sin(\alpha-\beta)+\sin(\alpha+\beta)); \tag{1.36}$$

$$\cos\alpha\cdot\cos\beta=\frac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta)); \tag{1.37}$$

$$\sin\alpha\cdot\sin\beta=\frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta)). \tag{1.38}$$

**Example 1.60.** Calculate $\cos 45^\circ\cdot\cos 15^\circ$.

*Solution*.

$$\cos 45^\circ\cdot\cos 15^\circ=\frac{1}{2}\left(\cos\left(60^\circ\right)+\cos\left(30^\circ\right)\right)=\frac{1}{2}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{3}+1}{4}.$$

**Example 1.61.** Transform the expression to a sum:

$$4\cos\frac{\alpha}{2}\cdot\cos\alpha\cdot\sin\frac{3\alpha}{2}.$$

*Solution*.

$$4\cos\frac{\alpha}{2}\cdot\cos\alpha\cdot\sin\frac{3\alpha}{2}=4\cdot\left(\cos\frac{\alpha}{2}\cdot\sin\frac{3\alpha}{2}\right)\cdot\cos\alpha=$$

$$=4\cdot\frac{1}{2}\left(\sin\left(\frac{3\alpha}{2}-\frac{\alpha}{2}\right)+\sin\left(\frac{3\alpha}{2}+\frac{\alpha}{2}\right)\right)\cdot\cos\alpha=2(\sin\alpha+\sin 2\alpha)\cos\alpha=$$

$$= 2\sin\alpha\cos\alpha + 2\sin 2\alpha\cos\alpha = \sin 2\alpha + 2\cdot\frac{1}{2}(\sin 3\alpha + \sin\alpha) =$$

$$= \sin\alpha + \sin 2\alpha + \sin 3\alpha.$$

**Example 1.62.** Prove the identity:

$$\cos^2(45^\circ - \alpha) - \cos^2(60^\circ + \alpha) - \cos(75^\circ - 2\alpha) = \sin 2\alpha.$$

*Solution*. Let's transform the left part of the equality:

$$\cos^2(45^\circ - \alpha) - \cos^2(60^\circ + \alpha) - \cos(75^\circ - 2\alpha) =$$

$$= \frac{1+\cos(90^\circ - 2\alpha)}{2} - \frac{1+\cos(120^\circ + 2\alpha)}{2} -$$

$$- \frac{\sin(75^\circ + 75^\circ - 2\alpha) + \sin(75^\circ - 2\alpha - 75^\circ)}{2} =$$

$$= \frac{1}{2}\left(\sin 2\alpha - \cos(90^\circ + (30^\circ + 2\alpha)) - \sin(180^\circ - (30^\circ + 2\alpha)) + \sin 2\alpha\right) =$$

$$= \frac{1}{2}\left(\sin 2\alpha + \sin(30^\circ + 2\alpha) - \sin(30^\circ + 2\alpha) + \sin 2\alpha\right) = \sin 2\alpha.$$

which was to be proved.

## Tasks for individual work

**Task 1.63.** Transform the expressions to a product:

1) $\cos 152^\circ + \cos 28^\circ$;

2) $\cos 48^\circ - \cos 12^\circ$;

3) $\cos 20^\circ - \sin 20^\circ$;

4) $\dfrac{\sin 75^\circ + \sin 15^\circ}{\cos 75^\circ - \sin 15^\circ}$;

5) $\sin\alpha + \sin 2\alpha + \sin 4\alpha + \sin 5\alpha$;

6) $\dfrac{\cos 6\alpha - \cos 4\alpha}{\cos 6\alpha + \cos 4\alpha}$;

7) $\dfrac{\sin 6\alpha - \sin 4\alpha}{\sin 6\alpha + \sin 4\alpha}$;

8) $\dfrac{1 - 2\cos\alpha + \cos 2\alpha}{1 + 2\cos\alpha + \cos 2\alpha}$;

9) $\cos 22^\circ + \cos 24^\circ + \cos 26^\circ + \cos 28^\circ$;

10) $\operatorname{tg} 9^\circ - \operatorname{tg} 27^\circ - \cos 63^\circ + \operatorname{tg} 81^\circ$;

11) $1 + 2\cos\alpha$;

12) $\sqrt{2} + 2\sin\alpha$;

13) $\sin\alpha + \sqrt{3}\cos\alpha$.

**Task 1.64.** Transform the expressions to a sum:

1) $4\cos\frac{\alpha}{2}\cos\alpha\sin\frac{5\alpha}{2}$; 2) $4\cos\frac{\alpha}{2}\cos\alpha\cos\frac{5\alpha}{2}$;

3) $\sin\alpha\cos 2\alpha\cos 3\alpha$; 4) $\cos 7\alpha\cos 3\alpha - \cos 8\alpha\cos 2\alpha$;

5) $\sin\left(\frac{\pi}{4} + \alpha\right)\sin\left(\frac{\pi}{4} - \alpha\right)$; 6) $4\cos\left(\frac{\pi}{12} - \alpha\right)\sin\left(\frac{\pi}{12} + \alpha\right)$.

**Task 1.65.** Simplify the expressions:

1) $\frac{\sin\alpha + 2\sin 3\alpha + \sin 5\alpha}{\sin 3\alpha + 2\sin 5\alpha + \sin 7\alpha}$; 2) $\frac{1 + \sin 4\alpha - \cos 4\alpha}{1 + \sin 4\alpha + \cos 4\alpha}$;

3) $\frac{\sin 2\alpha - \sin 3\alpha + \sin 4\alpha}{\cos 2 - \cos 3\alpha + \cos 4\alpha}$; 4) $\frac{\sin 2\alpha - \sin 6\alpha + \cos 2\alpha - \cos 6\alpha}{\sin 4\alpha - \cos 4\alpha}$;

5) $\frac{\cos\left(2\alpha - \frac{\pi}{2}\right) + \sin(3\pi - 4\alpha) - \cos\left(\frac{\pi}{2} + 6\alpha\right)}{4\sin(5\pi - 3\alpha)\cdot\cos(\alpha - 2\pi)}$;

6) $\frac{\sin^4\alpha - \cos^2\alpha + \cos^2\alpha}{2(1 - \cos\alpha)}$;

7) $1 - \cos(2\alpha - \pi) - \cos(4\alpha + \pi) + \cos(6\alpha - 2\pi)$;

8) $4\sin\left(\frac{\pi}{3} - \alpha\right)\sin\alpha\sin\left(\frac{\pi}{3} + \alpha\right)$.

**Task 1.66.** Prove the identities:

1) $\frac{\cos 6\alpha - \cos 10\alpha}{\sin 8\alpha} = 2\sin 2\alpha$;

2) $\frac{(\sin 2\alpha + \sin 6\alpha)(\cos 2\alpha - \cos 6\alpha)}{1 - \cos 8\alpha} = \sin 4\alpha$;

3) $\frac{\sin^2 4\alpha}{2\cos\alpha + \cos 3\alpha + \cos 5\alpha} = 2\sin\alpha\cdot\sin 2\alpha$;

4) $\sin\alpha + 2\sin 3\alpha + \sin 5\alpha = 4\sin 3\alpha \cdot \cos^2\alpha$;

5) $\sin 2\alpha + \sin 4\alpha - \sin 6\alpha = 4\sin\alpha \cdot \sin 2\alpha \cdot \sin 3\alpha$;

6) $\dfrac{2\cos^2 2\alpha + \sqrt{3}\sin 4\alpha - 1}{2\sin^2 2\alpha + \sqrt{3}\sin 4\alpha - 1} = \dfrac{\sin(4\alpha + 30^\circ)}{\sin(4\alpha - 30^\circ)}$.

## 1.10. The simplest trigonometric equations

The simplest trigonometric equations are equations like $\sin x = a$, $\cos x = a$, $\operatorname{tg} x = a$, $\operatorname{ctg} x = a$. Let's consider each equation.

1. The equation $\sin x = a$. If $|a| > 1$, this equation doesn't have any solutions. If $|a| \le 1$, this equation has an infinite set of solutions, which are defined using the following formula:

$$x = (-1)^n \arcsin a + \pi n,\ n \in Z. \tag{1.39}$$

In this formula the value $\arcsin a$ (arcsine of $a$) means an angle such that its sine equals $a$, where $-1 \le a \le 1$. For example, $\arcsin 1 = \dfrac{\pi}{2}$.

The properties of $\arcsin a$:

$$-\frac{\pi}{2} \le \arcsin a \le \frac{\pi}{2};\quad \arcsin(-a) = -\arcsin a.$$

Let's remember the basic values of $\arcsin a$:

$\arcsin\dfrac{1}{2} = \dfrac{\pi}{6}$; $\arcsin\left(-\dfrac{1}{2}\right) = -\dfrac{\pi}{6}$; $\arcsin\dfrac{\sqrt{2}}{2} = \dfrac{\pi}{4}$; $\arcsin\left(-\dfrac{\sqrt{2}}{2}\right) = -\dfrac{\pi}{4}$;

$\arcsin\dfrac{\sqrt{3}}{2} = \dfrac{\pi}{3}$; $\arcsin\left(-\dfrac{\sqrt{3}}{2}\right) = -\dfrac{\pi}{3}$.

Some particular cases of solving the equation $\sin x = a$:

$$\sin x = 0 \qquad x = \pi n,\ n \in Z; \tag{1.40}$$

$$\sin x = 1 \qquad x = \frac{\pi}{2} + 2\pi n,\ n \in Z; \tag{1.41}$$

$$\sin x = -1 \qquad x = -\frac{\pi}{2} + 2\pi n,\ n \in Z. \tag{1.42}$$

**Example 1.67.** Solve the following equations:

a) $\sin\left(x - \frac{\pi}{4}\right) = -1$; b) $\sin 2x = -\frac{\sqrt{3}}{2}$.

*Solution.* a) using (1.42) we have:

$$x - \frac{\pi}{4} = -\frac{\pi}{2} + 2\pi n;\ \ x = -\frac{\pi}{4} + 2\pi n,\ \ n \in Z;$$

b) using formula (1.39) we get:

$$2x = (-1)^n \arcsin\left(-\frac{\sqrt{3}}{2}\right) + \pi n = (-1)^n\left(-\frac{\pi}{3}\right) + \pi n;$$

$$2x = (-1)^{n+1}\frac{\pi}{3} + \pi n,$$

whence $x = (-1)^{n+1}\frac{\pi}{6} + \frac{\pi n}{2}, n \in Z$.

2. The equation $\cos x = a$. If $|a| > 1$, this equation doesn't have any solutions. If $|a| \le 1$, this equation has an infinite set of solutions, which are defined using the following formula:

$$x = \pm\arccos a + 2\pi n,\ n \in Z. \tag{1.43}$$

Here, the value $\arccos a$ (arccosine of $a$) means an angle such that its cosine equals $a$, where $-1 \le a \le 1$. The basic properties are: $0 \le \arccos a \le \pi$, $\arccos(-a) = \pi - \arccos a$.

For example, $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \pi - \arccos\frac{\sqrt{3}}{2} = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Let's remember the basic values of $\arccos a$:

$$\arccos\frac{\sqrt{3}}{2} = \frac{\pi}{6};\quad \arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6};$$

$$\arccos\frac{\sqrt{2}}{2}=\frac{\pi}{4};\ \ \arccos\left(-\frac{\sqrt{2}}{2}\right)=\frac{3\pi}{4};$$

$$\arccos\frac{1}{2}=\frac{\pi}{3};\ \ \arccos\left(-\frac{1}{2}\right)=\frac{2\pi}{3}.$$

Some particular cases of solving the equation $\cos x = a$:

$$\cos x = 0 \qquad x=\frac{\pi}{2}+2\pi n,\ n\in Z; \tag{1.44}$$

$$\cos x = 1 \qquad x = 2\pi n,\ n\in Z; \tag{1.45}$$

$$\cos x = -1 \qquad x=\pi+2\pi n,\ n\in Z. \tag{1.46}$$

**Example 1.68.** Solve the equations:

a) $2\cos 2x = -1$; b) $\sin\left(3x-\frac{\pi}{6}\right)=0.$

*Solution*. a) Let's rewrite the equation like

$$\cos 2x = -\frac{1}{2}.$$

Whence $x=\pm\arccos\left(-\frac{1}{2}\right)+2\pi n=\pm\frac{2\pi}{3}+2\pi n,\ n\in Z.$

Then $x=\pm\frac{\pi}{3}+2\pi n,\ n\in Z.$

b) $\cos\left(3x-\frac{\pi}{6}\right)=0$ is a particular case: $3x-\frac{\pi}{6}=\frac{\pi}{2}+2\pi n,$

$3x=\frac{\pi}{6}+\frac{\pi}{2}+\pi n=\frac{\pi}{3}+\pi n,$ and $x=\frac{\pi}{9}+\frac{\pi n}{3},n\in Z.$

3. The equations $\operatorname{tg} x = a$ and $\operatorname{ctg} x = a$. For any value of $a$ the equations have an infinite set of solutions, which are defined by the formulas:

$$\text{a) } x=\pm\operatorname{arctg} a+\pi n,\ n\in Z. \tag{1.47}$$

$$-\frac{\pi}{2}\le\operatorname{arctg} a\le\frac{\pi}{2},\ \ \operatorname{arctg}(-a)=-\operatorname{arctg} a,\ \operatorname{arctg}(0)=\frac{\pi}{2},\ \operatorname{arctg}(1)=\frac{\pi}{4},$$

$$\operatorname{arctg}\left(\sqrt{3}\right)=\frac{\pi}{3},\quad \operatorname{arctg}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}.$$

Some particular cases of solving the equations $\operatorname{tg} x = a$ and $\operatorname{ctg} x = a$:

$$\operatorname{tg} x = 0 \qquad x = \pi n,\ n \in Z;$$

$$\operatorname{tg} x = 1 \qquad x = \frac{\pi}{4} + \pi n,\ n \in Z;$$

$$\operatorname{tg} x = -1 \qquad x = -\frac{\pi}{4} + \pi n,\ n \in Z;$$

$$\operatorname{ctg} x = 0 \qquad x = \frac{\pi}{2} + \pi n,\ n \in Z;$$

$$\operatorname{ctg} x = 1 \qquad x = \frac{\pi}{4} + \pi n,\ n \in Z;$$

$$\operatorname{ctg} x = -1 \qquad x = \frac{3\pi}{4} + \pi n,\ n \in Z.$$

**Example 1.69.** Solve the following equations:

a) $3\operatorname{tg}\left(x-\frac{\pi}{3}\right)=\sqrt{3}$; b) $\operatorname{ctg} 4x = 3$.

*Solution*. a) We have $\operatorname{tg}\left(x-\frac{\pi}{3}\right)=\frac{\sqrt{3}}{3}$, whence $x-\frac{\pi}{3}=\operatorname{arctg}\frac{\sqrt{3}}{3}+\pi n$,

$x-\frac{\pi}{3}=\frac{\pi}{6}+\pi n=\frac{\pi}{2}+\pi n, n\in Z$;

b) $\operatorname{ctg} 4x = 3$, $4x = \operatorname{arcctg} 3 + \pi n$, $x=\frac{1}{4}\operatorname{arcctg} 3+\pi n$, $n \in Z$.

## Tasks for individual work

**Task 1.70.** Calculate:

1) $\arcsin\left(-\frac{\sqrt{3}}{2}\right)+2\arccos\left(-\frac{\sqrt{3}}{2}\right)$;

2) $\arccos\left(-\frac{1}{2}\right)+2\operatorname{arctg}\left(-\frac{1}{\sqrt{3}}\right)$;

3) $5\arccos(-1)-12\arccos\left(\frac{\sqrt{3}}{2}\right)-6\arccos\left(\frac{\sqrt{2}}{2}\right)$;

4) $\operatorname{arctg}(-1)+\operatorname{arctg}(-\sqrt{3})-\operatorname{arctg}\left(-\frac{1}{\sqrt{3}}\right)-\operatorname{arctg}0$;

5) $2\arccos(-1)-4\arccos\left(\frac{\sqrt{2}}{2}\right)+\operatorname{arctg}\left(\frac{\sqrt{3}}{3}\right)-\operatorname{arctg}1$.

**Task 1.71.** Calculate the following expressions:

1) $\cos\left(2\cos\frac{\sqrt{2}}{2}\right)$; 2) $\operatorname{tg}\left(\arccos\frac{1}{2}\right)$; 3) $\cos\left(2\arcsin\left(-\frac{\sqrt{2}}{2}\right)\right)$;

4) $\sin\left(\arcsin\frac{1}{2}+\arccos\frac{1}{2}\right)$; 5) $\cos\left(\arccos\left(-\frac{\sqrt{3}}{2}\right)+\frac{1}{2}\arcsin\frac{\sqrt{3}}{2}\right)$.

**Task 1.72.** Solve the simplest trigonometric equations:

1) $\sin 5x=1$; 2) $2\sin 10x-\sqrt{3}=0$; 3) $2\sqrt{3}\cos 10x+3=0$;

4) $\cos\left(x+\frac{\pi}{6}\right)=0$; 5) $\operatorname{tg}\left(2x-\frac{\pi}{6}\right)=0$; 6) $\sin\left(2x-\frac{\pi}{4}\right)=-1$;

7) $\sqrt{3}\operatorname{tg}\left(3x+\frac{\pi}{6}\right)=1$.

## 1.11. Methods for solving trigonometric equations

When solving more difficult trigonometric equations we take into account that there is no single method for solving them. As a rule, such equations can be reduced to the simplest equations using the substitution of variables.

1. Solving the equations reduced to quadratic equations.

**Example 1.73.** Solve the following equations:

a) $2\cos^2 x+5\sin x-4=0$; b) $\operatorname{tg} x+5\operatorname{ctg} x=6$.

*Solution*. a) Let's substitute $1-\sin^2 x$ for $\cos^2 x$ and get the equation with one trigonometric function $\sin x$:

$$2(1-\sin^2 x)+\sin x-4=0 \text{ or } 2\sin^2 x-5\sin x+2=0.$$

Let's substitute $\sin x=t$, where $|t|\le 1$: $2t^2-5t+2=0$, whence $t_1=\frac{1}{2}$, $t_2=2$. The root $t_2=2$ doesn't satisfy the condition $|t|\le 1$. $\sin x=\frac{1}{2}$, whence $x=(-1)^n\frac{\pi}{6}+\pi n$, $n\in Z$.

b) In this equation, $\text{tg } x+5\text{ctg } x=6$. Let's substitute $\frac{1}{\text{tg } x}$ for $\text{ctg } x$: $\text{tg } x+\frac{5}{\text{tg } x}=6$ or $\text{tg}^2 x-6\text{tg } x+5=0$, where $\text{tg } x\ne 0$, i.e. $x\ne \pi n$. Let's substitute $\text{tg } x=t$. Then $t^2-6t+5=0$, whence $t_1=1$, $t_2=5$.

We have two simplest equations:

1) $\text{tg } x=1$, $x=\frac{\pi}{4}+\pi n$, $n\in Z$;

2) $\text{tg } x=5$, $x=\text{arctg } 5+\pi n$, $n\in Z$.

2. Solving homogeneous trigonometric equations and equations reduced to them. Let's consider some homogeneous equations of the first and second orders:

a) $a\cdot\cos x+b\cdot\sin x=0$;

b) $a\cdot\cos^2 x+b\cdot\cos x\cdot\sin x+c\cdot\sin^2 x=0$.

Such equations can be solved by dividing them by $\cos x\ne 0$ (in the case a) and by $\cos^2 x\ne 0$ (in the case b). We don't lose roots, because $\cos x=0$ isn't the root of these equations.

**Example 1.74.** Solve the following equations:

a) $\sqrt{3}\cdot\sin 2x+\cos 2x=0$;

b) $3\cdot\sin^2 x-2\cdot\cos x\cdot\sin x-\cos^2 x=0$;

c) $4\cdot\sin^2 x-4\cdot\sin 2x+10\cos^2 x=3$.

*Solution*. a) $\sqrt{3}\cdot\sin 2x+\cos 2x=0$. It's a homogeneous equation of the first order. Let's divide all the summands by $\cos 2x$ and get:

$\sqrt{3}\cdot \operatorname{tg} 2x+1=0$, whence $\operatorname{tg} 2x=-\frac{1}{\sqrt{3}}$, and $2x=\operatorname{arctg}\left(-\frac{1}{\sqrt{3}}\right)+\pi n$, or $2x=-\frac{\pi}{6}+\pi n$, $x=-\frac{\pi}{12}+\frac{\pi n}{2}$, $n\in Z$.

b) $3\cdot\sin^2 x-2\cdot\cos x\cdot\sin x-\cos^2 x=0$. It's a homogeneous equation of the second order. Let's divide all the summands by $\cos^2 x$ and get: $3\operatorname{tg}^2 x-2\operatorname{tg} x-1=0$.

Let's substitute $\operatorname{tg} x=t$ into the equation. Then $3t^2-2t-1=0$, whence $t_1=1$, $t_2=-\frac{1}{3}$. Let's solve two simplest equations:

1) $\operatorname{tg} x=1$, $x=\frac{\pi}{4}+\pi n$;

2) $\operatorname{tg} x=-\frac{1}{3}$, $x=-\operatorname{arctg}\frac{1}{3}+\pi n$, $n\in Z$.

c) $4\cdot\sin^2 x-4\cdot\sin 2x+10\cos^2 x=3$.

Let's transform this equation to a homogeneous equation:

$$4\cdot\sin^2 x-8\cdot\sin x\cdot\cos x+10\cos^2 x=3\cdot\left(\sin^2 x+\cos^2 x\right),$$

or $4\cdot\sin^2 x-8\cdot\sin x\cdot\cos x+7\cos^2 x=0$.

Let's divide it by $\cos^2 x$. If $\operatorname{tg} x=t$, then $t^2-8t+7=0$, whence $t_1=1$, $t_2=7$. Let's get back to the variable $x$:

1) $\operatorname{tg} x=1$, $x=\frac{\pi}{4}+\pi n$, $n\in Z$; 2) $\operatorname{tg} x=7$, $x=\operatorname{arctg} 7+\pi n$, $n\in Z$.

3. Solving equations of the form $f(x)=0$ using the transformation of a sum of functions to a product. It's one out of the ways of factorizing an expression into factors of the left part of the equation $f(x)=0$. It allows us to equate each factor to zero and reduce the equation solving the given equation to a set of more simple or simplest equations.

**Example 1.75.** Solve the equations:

a) $\cos 7x + \cos 3x = 0$;

b) $\sin x + \sin 2x + \sin 3x = 0$;

c) $\cos 2x - \cos 8x + \cos 6x = 1$.

*Solution*. a) $\cos 7x + \cos 3x = 0$.

Let's transform the left part of the equation to the product: $2\cos 5x \cdot \cos 2x = 0$, whence we get $\cos 5x = 0$ or $\cos 2x = 0$; i.e.

$$5x = \frac{\pi}{2} + \pi n \text{, and } x_1 = \frac{\pi}{10} + \frac{\pi n}{5}, n \in Z.$$

$2x = \frac{\pi}{2} + \pi n$, and $x_2 = \frac{\pi n}{4} + \frac{\pi n}{2}, n \in Z$.

b) $\sin x + \sin 2x + \sin 3x = 0$.

Let's transform the left part of the equation to a product:

$$(\sin 3x + \sin x) + \sin 2x = 0;$$

$$(2\sin 2x \cdot \cos x) + \sin 2x = 0;$$

$$\sin 2x \cdot (2\cos x + 1) = 0;$$

$$\sin 2x = 0 \text{ or } 2\cos x + 1 = 0.$$

From the first equation we get: $2x = \pi n,\ x = \frac{\pi n}{2},\ n \in Z$.

From the second equation: $\cos x = -\frac{1}{2},\ x = \pm\frac{2\pi}{3} + 2\pi n,\ n \in Z$.

c) $\cos 2x - \cos 8x + \cos 6x - 1 = 0$.

Let's group the summands and transform each sum to a product:

$$(\cos 6x + \cos 2x) - (1 + \cos 8x) = 0,$$

$$2\cos 4x \cdot \cos 2x - 2\cos^2 4x = 0,$$

$$2\cos 4x \cdot (\cos 2x - \cos 4x) = 0,$$

$$2\cos 4x \cdot (\sin 3x \cdot \sin x) = 0.$$

Then we get three simplest equations:

$\cos 4x = 0, \quad 4x = \frac{\pi}{2} + \pi n, \quad x = \frac{\pi}{8} + \frac{\pi n}{4}, n \in Z;$

$\sin 3x = 0, \quad 3x = \pi n, \quad x = \frac{\pi n}{3}, n \in Z;$

$\sin x = 0, \quad x = \pi n, \quad n \in Z.$

The solution $x = \pi n$ is included into the solution $x = \frac{\pi n}{3}$. Therefore the solution to the given equation will be $x = \frac{\pi}{8} + \frac{\pi n}{4}$ or $x = \frac{\pi n}{3}$, $n \in Z$.

4. Solving trigonometric equations using the transformation of a product of trigonometric functions to a sum.

**Example 1.76.** Solve the following equation:

$$\sin 2x \cdot \sin 6x - 3\cos\frac{\pi}{2} = \cos x \cdot \cos 3x.$$

*Solution*. Let's transform the left and right sides of the equation to a sum:

$$\frac{1}{2}[\cos(6x - 2x) - \cos(2x + 6x)] - 3 \cdot 0 = \frac{1}{2}[\cos(x + 3x) + \cos(3x - x)];$$

$$\cos 4x - \cos 8x = \cos 4x + \cos 2x; \text{ or } \cos 8x + \cos 2x = 0.$$

Let's transform this sum to a product: $2\cos 5x \cdot \cos 3x = 0,$ whence

1) $\cos 5x = 0, \quad 5x = \frac{\pi}{2} + \pi n, \quad x = \frac{\pi}{10} + \frac{\pi n}{5}, n \in Z;$

2) $\cos 3x = 0, \quad 3x = \frac{\pi}{2} + \pi n, \quad x = \frac{\pi}{6} + \frac{\pi n}{3}, n \in Z.$

5. Solving trigonometric equations using the formulas of reducing a power. The formulas of reducing a power have the form:

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

**Example 1.77.** Solve the equation:

$$\sin x \cdot \sin 2x + \cos^2 x = \sin 4x \cdot \sin 5x + \cos^2 4x.$$

*Solution*. Let's apply the formulas of the transformation of a product of functions to a sum and the formulas of reducing a power:

$$\frac{1}{2}(\cos x+\cos 3x)+\frac{1+\cos 2x}{2}=\frac{1}{2}(\cos x-\cos 9x)+\frac{1+\cos 8x}{2}.$$

After the simplification we get:

$$(\cos 9x-\cos 3x)+(\cos 2x-\cos 8x)=0.$$

Let's transform this sum into a product:

$$-2\sin 3x\cdot\sin 6x+2\sin 5x\cdot\sin 3x=0 \text{ or}$$

$$\sin 3x\cdot(\sin 6x-\sin 5x)=0,$$

$$2\sin 3x\cdot\sin\frac{x}{2}\cdot\cos\frac{11x}{2}=0.$$

Then we get three simplest equations:

$$\sin 3x=0,\quad 3x=\pi n,\quad x=\frac{\pi n}{3},\ n\in Z;$$

$$\sin\frac{x}{2}=0,\quad \frac{x}{2}=\pi n,\quad x=2\pi n,\ n\in Z;$$

$$\cos\frac{11x}{2}=0,\quad \frac{11x}{2}=\frac{\pi}{2}+\pi n,\quad x=\frac{\pi}{11}+\frac{2\pi n}{11},\ n\in Z.$$

6. Solving trigonometric equations using the formulas:

$$\sin x=\frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}},\qquad \cos x=\frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}.$$

Let's note that the use of these formulas can be reduced to the loss of a solution. It's known that $\operatorname{tg}\frac{x}{2}$ makes sense for $\frac{x}{2}\neq\frac{\pi}{2}+\pi n$, i.e. for $x\neq\pi+2\pi n$, therefore it is necessary to check whether these are solutions or not, using the substitution of values $x=\pi+2\pi n$ into the given equation.

**Example 1.78.** Solve the equation: $3\sin x+2\cos x=3$.

*Solution*. Let's express $\sin x$ and $\cos x$ through $\operatorname{tg}\frac{x}{2}$:

$$\frac{6\,\mathrm{tg}\frac{x}{2}}{1+\mathrm{tg}^2\frac{x}{2}}+2\cdot\frac{1-\mathrm{tg}^2\frac{x}{2}}{1+\mathrm{tg}^2\frac{x}{2}}=3.$$

After the transformations we get the following equation:

$$5\,\mathrm{tg}^2\frac{x}{2}-6\,\mathrm{tg}\frac{x}{2}+1=0.$$

Let's substitute $\mathrm{tg}\frac{x}{2}=t$ into the equation. Then $5t^2-6t+1=0$, $t_1=1$; $t_2=\frac{1}{5}$. We get: 1) $\mathrm{tg}\frac{x}{2}=1$ $\frac{x}{2}=\frac{\pi}{4}+\pi n,$ $x=\frac{\pi}{2}+2\pi n, n\in Z.$

2) $\mathrm{tg}\frac{x}{2}=\frac{1}{5}$, $\frac{x}{2}=\mathrm{arctg}\frac{1}{5}+\pi n,$ $x=2\,\mathrm{arctg}\frac{1}{5}+2\pi n, n\in Z.$

Then we should check whether there is a loss of roots or not. An unused formula is correct if $\frac{x}{2}\neq\frac{\pi}{2}+\pi n,$ i.e. $x\neq\pi+2\pi n$. Let's check whether these values $x=\pi+2\pi n$ are roots of the given equation:

$$3\sin(\pi+2\pi n)+2\cos(\pi+2\pi n)=3, \ \Rightarrow 3\sin\pi+2\cos\pi=3,$$

or $5-2=3$, i.e. $x=\pi+2\pi n$ are roots of the given equation.

7*. Solving trigonometric equations including $\sin x\pm\cos x$ or $\sin x\cdot\cos x$. Such equations can be solved with the help of the substitution of $\sin x\pm\cos x=t$ into the equation. Then, raising both parts of the equation to the second power, we find $\sin x\cdot\cos x$.

**Example 1.79*.** Solve the following equation:

$$12-12(\sin x+\cos x)+5\sin 2x=0.$$

*Solution.* Let's substitute $\sin x+\cos x=t$ into the equation and raise it to the second power:

$$\sin^2 x+\cos^2 x+2\sin x\cdot\cos x=t^2,\ 1+\sin 2x=t^2,\ \sin 2x=t^2-1.$$

The equation takes the form $12-12t+5(t^2-1)=0$ or $5t^2-12t+7=0$, whence $t_1=1,\quad t_2=\dfrac{7}{5}.$

The given equation is reduced to a set of two equations:

1) $\sin x+\cos x=1\ \Rightarrow\ \sin x=1-\cos x,$

$$2\sin\frac{x}{2}\cos\frac{x}{2}=2\sin^2\frac{x}{2}\Rightarrow\sin\frac{x}{2}\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)=0.$$

a) $\sin\dfrac{x}{2}=0\quad\dfrac{x}{2}=\pi n\Rightarrow x=2\pi n, n\in Z.$

b) $\cos\dfrac{x}{2}-\sin\dfrac{x}{2}=0,\Rightarrow 1-\operatorname{tg}\dfrac{x}{2}=0\Rightarrow\operatorname{tg}\dfrac{x}{2}=1,$

$$\frac{x}{2}=\frac{\pi}{4}+\pi n\Rightarrow x=\frac{\pi}{2}+2\pi n, n\in Z.$$

2) $\sin x+\cos x=\dfrac{7}{5},\ \Rightarrow\ \sqrt{2}\sin\left(x+\dfrac{\pi}{4}\right)=\dfrac{7}{5},\quad \sin\left(x+\dfrac{\pi}{4}\right)=\dfrac{7}{5\sqrt{2}},$

$$x+\frac{\pi}{4}=(-1)^n\arcsin\frac{7}{5\sqrt{2}}+\pi n\Rightarrow x=(-1)^n\arcsin\frac{7}{5\sqrt{2}}-\frac{\pi}{4}+\pi n, n\in Z.$$

## Tasks for individual work

Solve the following equations:

**Task 1.80.** 1) $2\sin^2 x-3\sin x+1=0;$

2) $2\cos^2 x-\sin x+1=0;$

3) $\operatorname{tg}^2 2x-7\operatorname{tg}2x+10=0;$

4) $\cos^4 x-\sin^4 x=\sin x;$

5) $\cos x+2\cos 2x=1.$

**Task 1.81.** 1) $\sin x+\cos x=0;$

2) $2\cos^2 x+5\sin x\cos x-3\sin^2 x=0;$

3) $2\sin^2 x - 5\sin x\cos x - 8\cos^2 x + 2 = 0.$

**Task 1.82.** 1) $\cos 5x = \cos 3x;$

2) $\sin 2x = \sin 3x;$

3) $\cos x - \cos 2x = \sin 3x;$

4) $\cos x + \cos 3x + \cos 5x = 0;$

5) $\sin x - \sin 2x + \sin 3x - \sin 4x = 0;$

6) $\sqrt{3}\sin 2x + \cos 5x - \cos 9x = 0;$

7) $\cos 3x = \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = 0.$

**Task 1.83.** 1) $\cos x \cdot \cos 2x = \sin 7x \cdot \sin 6x;$

2) $2\cos x \cdot \sin 3x = \sin 4x + 1;$

3) $\sin 5x \cdot \cos 6x + \sin x = \sin 7x \cdot \cos 4x.$

**Task 1.84.** 1) $\cos^2 x + \sin^2 3x = 1;$

2) $\sin^2 x + \sin^2 2x = \sin^2 3x;$

3) $\sin^2 x + \sin^2 2x - \sin^2 3x - \sin^2 4x = 0;$

4) $3\sin\left(\frac{\pi}{2} - x\right) - 4 \cdot \sin(\pi + x) \cdot \sin\left(\frac{5\pi}{2} + x\right) + 8\cos^2\frac{x}{2} = 4.$

**Task 1.85.** 1) $\sin x - \cos x = 1;$ 2) $\sin x + 7\cos x = 5;$

3) $3\sin x - 2\cos x = 2;$ 4) $7\cos x - 8\sin x = 11.$

**Task 1.86.** 1) $\sin x + \sin x \cdot \cos x + \cos x = 1;$

2) $5\sin 2x + \sin x + \cos x = 1;$

3) $\sin x + \sin\left(\frac{3\pi}{2} + x\right) = 1 - 0.5\sin 2x.$

**Task 1.87.** Solve the equation using different methods of solving:

1) $\sin x + \cos 2x = 2;$ 2) $\cos x + 2\cos 2x = 1;$

3) $3 + 5\sin 2x = \cos 4x;$ 4) $\operatorname{tg} 3x - \operatorname{tg} x = 0;$

5) $\sin x + \cos x = \sqrt{2}\sin 5x;$ 6) $1 + 2\cos 3x \cdot \cos x = \cos 2x;$

7) $\sin 5x + \sin x + 2\sin^2 x = 1;$

8) $\sin 4x\left(1 + \sin\left(\frac{\pi}{2} - 4x\right)\right) = \cos^2(2x - \pi);$

9) $\sin^4 x+\cos^4 x=\sin 2x-0.5;$

10) $\cos\frac{x}{2}\cdot\cos\frac{3x}{2}-\sin x\cdot\sin 3x=\sin 2x\cdot\sin 3x;$

11) $1+\sin 2x=(\cos 3x+\sin 3x)^2;$

12) $4\cos^2 x=3\sin 2x-2\sin^2 x;$

13) $5(\sin x+\cos x)^2-12(\sin x+\cos x)+7=0;$

14) $4\sin^3 x+4\sin^2 x-3\sin x=3.$

## 1.12. The systems of trigonometric equations

The systems of trigonometric equations can be solved with the help of methods for solving algebraic systems.

**Example 1.88.** Solve the systems of the following equations:

a) $\begin{cases}\sin x+\cos y=1;\\ x+y=\frac{\pi}{2};\end{cases}$ b) $\begin{cases}\sin x\cdot\cos y=\frac{3}{4};\\ \sin y\cdot\cos x=\frac{1}{4}.\end{cases}$

*Solution*. a) From the second equation we have $x=\frac{\pi}{2}-y$ and substitute it into the first one. We get: $\sin\left(\frac{\pi}{2}-y\right)+\cos y=1,$ or $\cos y+\cos y=1; 2\cos y=1,\ \cos y=\frac{1}{2},$ a $y=\pm\frac{\pi}{3}+2\pi n, n\in Z.$

Let's find $x$. 1) If $y=\frac{\pi}{3}+2\pi n,$ then $x=\frac{\pi}{2}-\left(\frac{\pi}{3}+2\pi n\right)=\frac{\pi}{6}-2\pi n.$

2) If $y=-\frac{\pi}{3}+2\pi n,$ then $x=\frac{\pi}{2}-\left(-\frac{\pi}{3}+2\pi n\right)=\frac{5\pi}{6}-2\pi n.$

Thus, $\begin{cases}x=\frac{5\pi}{6}-2\pi n,\\ y=-\frac{\pi}{3}+2\pi n;\end{cases}$ or $\begin{cases}x=\frac{\pi}{6}-2\pi n,\\ y=\frac{\pi}{3}+2\pi n;\end{cases}$ $n\in Z.$

b) Let's add and subtract these equations:

$$\begin{cases} \sin x \cdot \cos y + \sin y \cos x = 1, \\ \sin x \cdot \cos y - \sin y \cos x = \dfrac{1}{2}, \end{cases} \text{ or } \begin{cases} \sin(x+y) = 1, \\ \sin(x-y) = \dfrac{1}{2}, \end{cases}$$

whence $\begin{cases} x+y = \dfrac{\pi}{2} + 2\pi n, n \in Z, \\ x-y = (-1)^l \dfrac{\pi}{6} + \pi l, l \in Z. \end{cases}$

The second equation with $l = 2k$ (even numbers) takes the form $x - y = \dfrac{\pi}{6} + 2\pi k,$ $k \in Z$, and with $l = 2k+1$ (odd numbers) gives $x - y = -\dfrac{\pi}{6} + 2\pi k,$ $k \in Z$ or $x - y = \dfrac{5\pi}{6} + 2\pi k, k \in Z.$

Then we have two algebraic systems:

$$\begin{cases} x+y = \dfrac{\pi}{2} + 2\pi n, \\ x-y = \dfrac{\pi}{6} + 2\pi k, \end{cases} \text{ or } \begin{cases} x+y = \dfrac{\pi}{2} + 2\pi n, \\ x-y = \dfrac{5\pi}{6} + 2\pi k. \end{cases}$$

Solving these systems using addition, we find:

$$\begin{cases} x = \dfrac{\pi}{3} + \pi(n+k); \\ y = \dfrac{\pi}{6} + \pi(n-k); \end{cases} \text{ or } \begin{cases} x = \dfrac{2\pi}{3} + \pi(n+k); & n \in Z \\ y = -\dfrac{\pi}{6} + \pi(n-k); & k \in Z. \end{cases}$$

## Tasks for individual work

**Task 1.89.** Solve the systems of equations:

1) $\begin{cases} \sin x + \sin y = 1; \\ x + y = \pi; \end{cases}$

2) $\begin{cases} \operatorname{tg} x + \operatorname{tg} y = 1; \\ x + y = \dfrac{\pi}{4}; \end{cases}$

3) $\begin{cases} \sin x \cdot \sin y = 0.25; \\ \cos x \cdot \cos y = 0.75; \end{cases}$

4) $\begin{cases} \cos x \cdot \cos y + \sin x \sin y = \dfrac{1}{2}; \\ \sin x \cdot \cos y - \sin y \cos x = 1. \end{cases}$

## Questions for self-assessment

1. What trigonometric function is odd?
2. What trigonometric function is even?
3. How is the sign of a trigonometric function defined?
4. What is a period of $\sin x$, $\cos x$?
5. What is a period of $\operatorname{tg} x$, $\operatorname{ctg} x$?
6. What is boundedness of trigonometric functions?
7. Name the basic trigonometric identities.
8. Name the basic theorems of addition.
9. How are reduction formulas for trigonometric functions used?
10. Name the basic formulas of double arguments.
11. Name the basic formulas of half arguments.
12. Name the basic formulas of triple arguments.
13. Name the basic formulas for transformation of a sum of trigonometric functions to a product.
14. Name the basic formulas for transformation of a product of trigonometric functions to a sum.
15. What is the solution to the trigonometric equation $\sin x = a$?
16. What is the solution to the trigonometric equation $\cos x = a$?
17. What is the solution to the trigonometric equation $\operatorname{tg} x = a$?
18. What is the solution to the trigonometric equation $\operatorname{ctg} x = a$?
19. Name the inverse trigonometric functions.
20. What is a homogeneous trigonometric equation?

# 2. Logarithmic and exponential functions

## 2.1. The logarithm

In mathematics, the logarithm is the inverse function to exponentiation. It follows from the equation $a^x = b$ , where $a > 0$ and $a \neq 1$, that $x$ is the exponent in which we have to raise the number $a$ (the base) to obtain the number $b$. This exponent is called the logarithm of a given number $b$ to the base $a$ and is denoted as

$$x = \log_a b.$$

For example:

$$\log_2 8 = 3; \log_3 \frac{1}{9} = -2; \log_8 \frac{1}{8} = -1.$$

If we substitute $x = \log_a b$ in the equation $a^x = b$, we obtain a very useful identity

$$a^{\log_a b} = b, \qquad (2.1)$$

which is called the main logarithmic identity.

The logarithm to base 10 (that is a = 10) is called the common (or decimal) logarithm and is denoted as $\lg b$. The main logarithmic identity for common logarithms may be written as

$$10^{\lg b} = b. \qquad (2.2)$$

For decimal logarithms, you should remember the values that are often used in solving a number of problems:

$$\lg 10 = 1; \lg 100 = 2; \lg 1000 = 3;$$
$$\lg 0.1 = -1; \lg 0.01 = -2; \lg 0.001 = -3, \ldots$$

## 2.2. Logarithm laws

1. Logarithms exist only for positive numbers, i.e. $\log_a b$ exists only for $a > 0$, $a \neq 1$ and $b > 0$.

2. The logarithm of a unit to any base $a$ ($a > 0$ and $a \neq 1$) equals zero, i.e.,

$$\log_a 1 = 0. \tag{2.3}$$

3. The logarithm of a base to the same number equals one,

$$\log_a a = 1. \tag{2.4}$$

4. The logarithm of a product is the sum of the logarithms of the factors,

$$\log_a(xy) = \log_a x + \log_a y, \quad (x > 0, y > 0), \tag{2.5}$$

$$\log_a(xy) = \log_a |x| + \log_a |y|, \quad (xy > 0). \tag{2.6}$$

5. The logarithm of a quotient of two numbers is the difference of the logarithms of the factors,

$$\log_a \frac{x}{y} = \log_a x - \log_a y, \quad (x > 0, y > 0), \tag{2.7}$$

$$\log_a \left(\frac{x}{y}\right) = \log_a |x| - \log_a |y|, \quad \left(\frac{x}{y} > 0\right). \tag{2.8}$$

6. The logarithm of the *n*-th power of a number is *n* times the logarithm of the number itself:

$$\log_a x^n = n \log_a x, \quad (x > 0), \tag{2.9}$$

$$\log_a x^{2n} = 2n \log_a |x|, \quad (x < 0). \tag{2.10}$$

Formulas (2.5 − 2.10) may be written in any direction.

7. The value of a logarithm will not change if the base and the number under the sign of the logarithm are raised to the same power:

$$\log_a x = \log_{a^n} x^n. \qquad (2.11)$$

8. The following formulas allow you to move to a new base:

$$\log_{a^\beta} x = \frac{1}{\beta}\log_a x, \qquad (2.12)$$

$$\log_{a^\beta} x^\alpha = \frac{\alpha}{\beta}\log_a x. \qquad (2.13)$$

9. The logarithm $\log_a x$ can be computed from the logarithm $\log_b x$ using the following formulas:

$$\log_a x = \frac{\log_b x}{\log_b a}, \qquad (2.14)$$

$$\log_a b = \frac{1}{\log_b a}. \qquad (2.15)$$

**Example 2.1.** Calculate using the properties of logarithms:

1) $2^{4+\log_2 3}$; 2) $27^{1-\log_3 2}$;

3) $\log_2 \log_2 \sqrt[8]{2}$;

4) $\sqrt{25^{\log_5 6} + 49^{\log_7 8}}$;

5) $\log_2 5 \cdot \log_{25} 8.$

*Solution.*

1) $2^{4+\log_2 3} = 2^4 \cdot 2^{\log_2 3} = 16 \cdot 3 = 48;$

2) $27^{1-\log_3 2} = \dfrac{27}{27^{\log_3 2}} = \dfrac{27}{27^{\log_{27} 8}} = \dfrac{27}{8};$

3) $\log_2 \log_2 \sqrt[8]{2} = \log_2\left(\dfrac{1}{8}\log_2 2\right) = \log_2 \dfrac{1}{8} = \log_2 2^{-3} = -3\log_2 2 = -3;$

4) $\sqrt{25^{\log_5 6}+49^{\log_7 8}}=\sqrt{25^{\log_{25} 36}+49^{\log_{49} 64}}=\sqrt{36+64}=\sqrt{100}=10;$

5) $\log_2 5\cdot\log_{25} 8=\log_2 5\cdot\log_{5^2} 2^3=\log_2 5\cdot\frac{3}{2}\log_5 2=\log_2 5\cdot\frac{3}{2}\log_5 2=\frac{3}{2}.$

**Example 2.2.** Calculate:

1) $\lg 56$, if $\lg 2=a,\ \log_2 7=b;$ 2) $\log_{12} 27$, if $\log_6 2=a.$

*Solution.*

1) $\lg 56=\lg(7\cdot 8)=\lg 7+\lg 8=\lg 7+3\lg 2=\dfrac{\log_2 7}{\log_2 10}+3\lg 2=$

$$=\frac{b}{\frac{1}{\lg 2}}+3a=ab+3a.$$

2) $\log_{12} 27=\dfrac{\log_6 27}{\log_6 12}=\dfrac{\log_6 3^3}{\log_6(2\cdot 6)}=\dfrac{3\log_6 3}{\log_6 2+\log_6 6}=\dfrac{3\log_6 \frac{6}{2}}{a+1}=$

$$=\frac{3\log_6 6-3\log_6 2}{a+1}=\frac{3(1-a)}{a+1}.$$

**Example 2.3.** Simplify:

$$A=(\log_a b+\log_b a+2)(\log_a b-\log_{ab} b).$$

*Solution.* Let's move on to the base $a$ in all terms using formulas (2.14) and (2.15):

$$A=\left(\log_a b+\frac{1}{\log_a b}+2\right)\left(\log_a b-\frac{\log_a b}{\log_a(ab)}\right)=$$

$$=\frac{\log_a^2 b+2\log_a b+1}{\log_a b}\left(\log_a b-\frac{\log_a b}{\log_a a+\log_a b}\right)=$$

$$=\frac{(\log_a b+1)^2}{\log_a b}\cdot\frac{\log_a b(1+\log_a b)-\log_a b}{1+\log_a b}=$$

$$=\frac{(\log_a b+1)}{\log_a b}\cdot\frac{\log_a b+\log_a^2 b-\log_a b}{}=(\log_a b+1)\cdot\log_a b.$$

## Tasks for individual work

**Task 2.4.** Calculate:

1) $2^{4+\log_2 3}$; 2) $5^{2\log_5 6}$; 3) $\sqrt{10^{2+\frac{1}{2}\lg 16}}$; 4) $-\log_2 \log_2 \sqrt[4]{\sqrt{2}}$;

5) $36^{\log_6 5} - 10^{1-\lg 2} - 3^{\log_9 36}$; 6) $49^{1-\log_7 2} + 5^{-\log_5 4}$;

7) $0.8\left(1+9^{\log_3 8}\right)^{\log_{65} 5}$; 8) $\log_2 36$, if $\log_{12} 9 = m$;

9) $\log_{30} 8$, if $\lg 5 = a$; $\lg 3 = b$.

## 2.3. The logarithmic function

The function $y = \log_a x$, ($a > 0$ and $a \neq 1$) is called a *logarithmic function*. We note further the main properties of this function.

1. The domain of this function is a set of all positive numbers, i.e.,

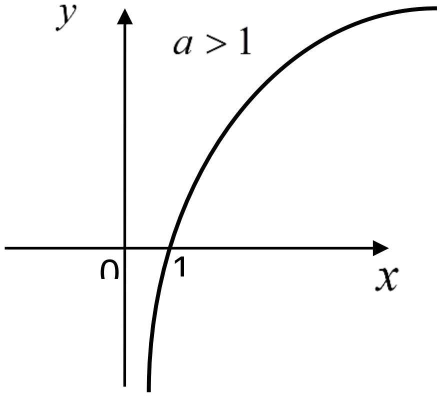
$$D(y) = (0;+\infty).$$

2. The codomain of this function is a set of all real numbers, i.e.,

$$E(y) = \Re = (-\infty;+\infty).$$

3. The function $y = \log_a x$ increases for $a > 1$ and decreases for $0 < a < 1$ .

4. If $x = 1$, then $\log_a x = 0$; i.e., its graph crosses the axes $Ox$ at the point $x = 1$. The function graph for $a > 1$ is shown in Fig. 2.1 and for $0 < a < 1$ in Fig. 2.2.



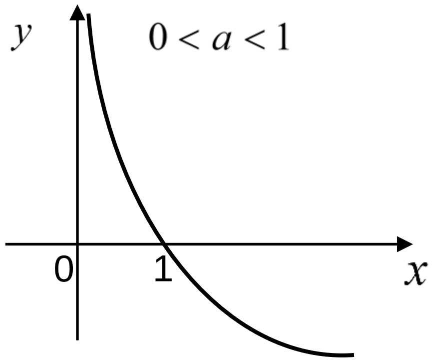



Fig. 2.1. **The base** $a>1$ Fig. 2.2. **The base** $0<a<1$

**Example 2.5.** Plot the graphs of the functions:

1) $y=\log_2(x+1)$; 2) $y=|\log_2 x|$;

3) $y=\sqrt{2^{\log_2 x^2}}$; 4) $y=3^{\log_3(1-x^2)}$.

*Solution.*

1) The domain of the function $y=\log_2(x+1)$ can be found from the condition $x+1>0$, wherefrom the domain is $x>-1$.

This function is increased in the domain. Let's also calculate $y$ in the point $x=1$: $y=\log_2(1+1)=1$. The graph is represented in Fig. 2.3.

2) Let us previously plot the graph of a more simple function $y=\log_2 x$, and then use the rule for plotting the function module $y=|f(x)|$ (Fig. 2.4).

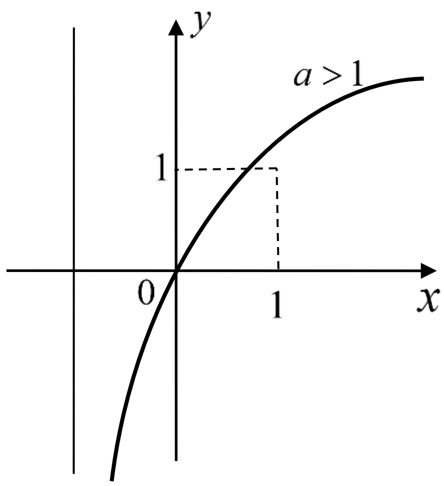



Fig. 2.3. **The graph of function 1)**

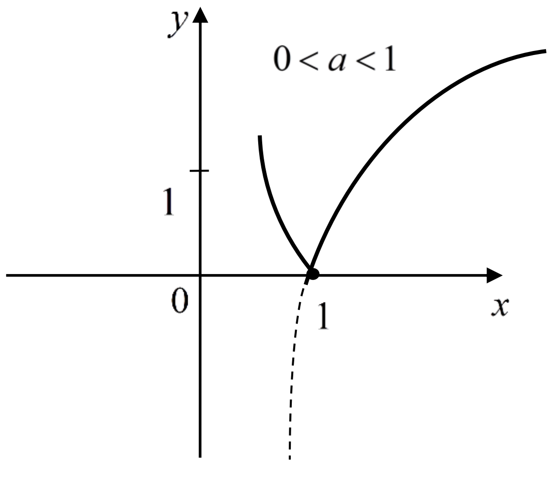



Fig. 2.4. **The graph of function 2)**

3) Let's simplify function 3) as follows:

$$y=\sqrt{2^{\log_2 x^2}}=\sqrt{x^2}=|x|; \quad x\neq 0.$$

The graph is represented in Fig. 2.5.

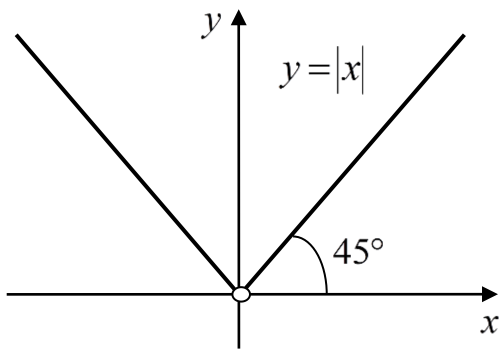



Fig. 2.5. **The graph of function 3)**

4) Let's simplify function 4) as follows:

$$y = 3^{\log_3\left(1-x^2\right)} = 1 - x^2, \quad 1 - x^2 > 0,$$

i.e., this function is a parabola. The graph is represented in Fig. 2.6.

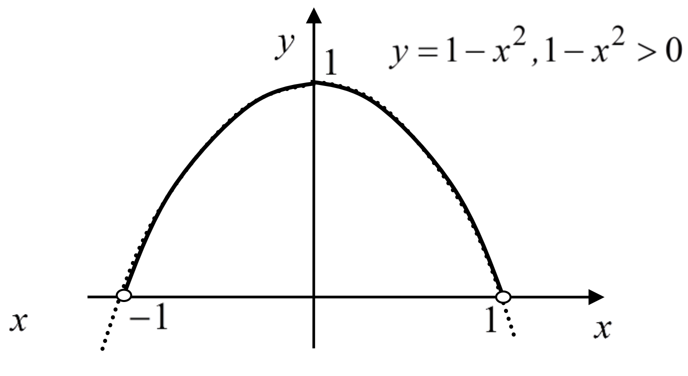



Fig. 2.6. **The graph of function 4)**

## Tasks for individual work

**Task 2.6.** Find the domains of the functions:

1) $y = \log_3(6 - 4x)$;

2) $y = \lg(x + 8) - 2\log_3(4 - x)$;

3) $y = \log_{\sqrt{2}} \dfrac{x-2}{x+2}$;

4) $y = \log_2(x^2 - 2x)$.

**Task 2.7.** Plot the graphs of the functions:

1) $y = \log_2(x-2)$;

2) $y = \log_2|x-2|$;

3) $y = 3^{\log_3(x-1)}$;

4) $y = \log_2 x + |\log_2 x|$.

## 2.4. The exponential function

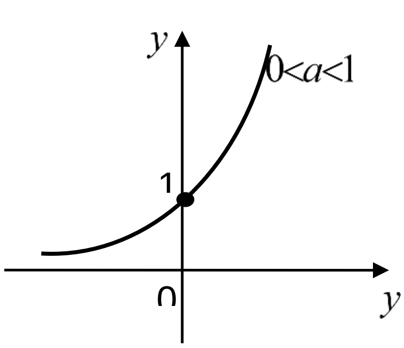
An exponential function is a function of the form

$$y = a^x, (a > 0, a \neq 1),$$

where $a$ is a positive real number, in which the argument $x$ occurs as an exponent.

### The main properties

1. The domain: $D(y) = (-\infty; +\infty)$.
2. The codomain: $E(y) = (0; +\infty)$.
3. If $x = 0$, $e^0 = 1$. I.e., the graph of the exponential function crosses the $Oy$ axis at the point $y = 1$.
4. If $a > 1$, the exponential function is increased (Fig. 2.7); if $0 < a < 1$, the exponential function is decreased (Fig. 2.8).



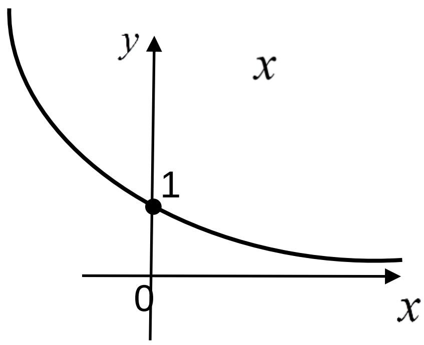



Fig. 2.7. **The graph of the exponential function if** $a>1$

Fig. 2.8. **The graph of the exponential function if** $0<a<1$

**Example 2.8.** Plot the graphs of the function:

$$1)\ y=2^{x};\ \ 2)\ y=2^{x-1};\ \ 3)\ y=2^{|x|};\ \ 4)\ y=\left(\frac{1}{2}\right)^{|x|}.$$

*Solution.*

1) The graph is depicted in Fig. 2.9.

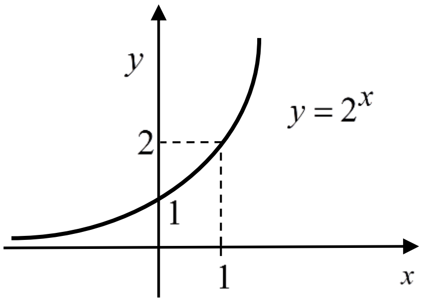



Fig. 2.9.**The function** $y=2^{x}$

2) To plot the graph of the function $y=2^{x-1}$ we can move the graph of the function $y=2^{x}$ to the right-hand side by 1 (Fig. 2.10).

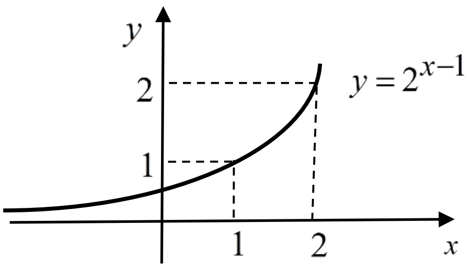



Fig. 2.10. **The function** $y=2^{x-1}$

3) We use the module definition and transform the function $y=2^{x}$ graph to obtain the function $y=2^{|x|}$ graph (Fig. 2.11).

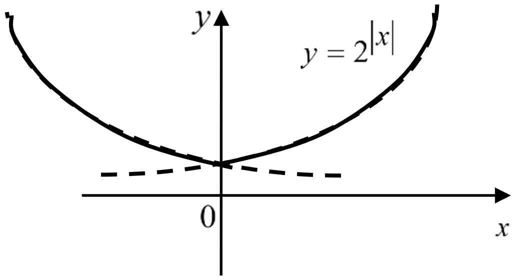



Fig. 2.11. **The function** $y = 2^{|x|}$

4) We use the modulus properties here:

$$x \geq 0,\ y = \left(\frac{1}{2}\right)^{x},\ \text{if}\ x < 0\ \ y = \left(\frac{1}{2}\right)^{-x} = 2^{x}.$$

Then we obtain the simple exponential functions: $y = 2^{-x}$ for $x \geq 0$ and $y = 2^{x}$ for $x < 0$, Fig. 2.12.

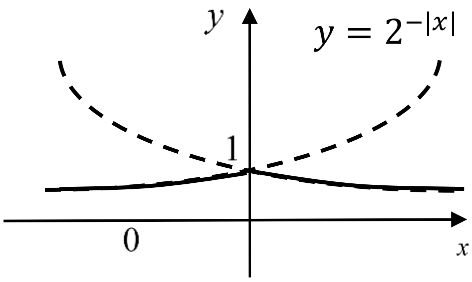



Fig. 2.12. **The function** $y = 2^{-|x|}$

## 2.5. Exponential equations

An exponential equation is the one in which a variable occurs in the exponent. The simplest exponential equation of a general form is the equation:

$$a^{f(x)} = b^{g(x)} \qquad \left(a > 0, b > 0, a \neq 1, b \neq 1\right), \qquad (2.16)$$

which is equivalent to the equation:

$$f(x) = \log_a b \cdot g(x). \qquad (2.17)$$

If $a^{f(x)} = a^{g(x)}$, then the solution is very simple: $f(x) = g(x)$.

We consider the methods of reducing exponential equations to more simple equations in the following examples.

**Example 2.9.** Solve the equations:

1) $2^x = 5$; 2) $2^{2x^2+x-6} = 1$; 3) $6^{3-x} = 216$;

4) $3^x \left(\frac{1}{3}\right)^{x-3} = \left(\frac{1}{27}\right)^x$; 5) $2^{x^2-3} \cdot 5^{x^2-3} = 0.01 \cdot \left(10^{x-1}\right)^3$.

*Solution.*

1) Using formula (2.17), we have: $x = \log_2 5$;

2) Let us transform both sides of the equation to base 2:

$$2^{2x^2+x-6} = 2^0 \;\Rightarrow\; 2x^2 + x - 6 = 0.$$

Then the solution is

$$x_1 = -2,\; x_1 = \frac{3}{2}.$$

3) From $6^{3-x} = 6^3$ we have $3 - x = 3$. Then the solution is $x = 0$.

4) Let's move on to base $\frac{1}{3}$:

$\left(\frac{1}{3}\right)^{-x} \cdot \left(\frac{1}{3}\right)^{x-3} = \left(\frac{1}{3}\right)^{3x}$, $\left(\frac{1}{3}\right)^{-x+x-3} = \left(\frac{1}{3}\right)^{3x}$, $3x = -3$, wherefrom $x = -1$.

5) Let's move on to base $10$:

$10^{x^2-3} = 10^{-2} \cdot 10^{3x-3}$, $10^{x^2-3} = 10^{-2+3x-3}$, $x^2 - 3 = 3x - 5$, $x^2 - 3x + 2 = 0$.
Then $x_1 = 1, x_2 = 2$.

**Example 2.10.** Solve the equations:

1) $2^{x+3}-2^{x}=112;$ 2) $5^{x}-5^{x-2}=24;$

3) $4\cdot 3^{x+2}+5\cdot 3^{x}-7\cdot 3^{x+1}=180;$ 4) $3^{x+1}-2\cdot 5^{x-2}=5^{x}-2\cdot 3^{x}.$

*Solution.*

1) We transform the equation using the property of powers $a^{m+n}=a^{m}\cdot a^{n}$:

$$2^{x}\cdot 2^{3}-2^{x}=112;\quad 2^{x}(8-1)=112;\quad 2^{x}\cdot 7=112;$$
$$2^{x}=16;\quad 2^{x}=2^{4};\quad x=4.$$

2) We use the property of powers ($a^{m-n}=\dfrac{a^{m}}{a^{n}}$):

$$5^{x}-\frac{5^{x}}{5^{2}}=24,\ \ 5^{x}\left(1-\frac{1}{5^{2}}\right)=24,\quad 5^{x}\,\frac{24}{25}=24;\quad 5^{x}=25,$$

wherefrom $x=2.$

3) We transform the equation:

$$4\cdot 3^{x}\cdot 3^{2}+5\cdot 3^{x}-7\cdot 3^{x}\cdot 3=180;\quad 3^{x}(36+5-21)=180;$$
$$3^{x}\cdot 20=180;\quad 3^{x}=9;\quad x=2.$$

4) We collect the powers with the same base in different parts of the equation:

$$3^{x+1}-2\cdot 5^{x-2}=5^{x}-2\cdot 3^{x},\ \ 3^{x}\cdot 3+2\cdot 3^{x}=5^{x}+2\cdot\frac{5^{x}}{5^{2}},$$
$$5\cdot 3^{x}=5^{x}\left(1+\frac{2}{5^{2}}\right),\ \ 5\cdot 3^{x}=5^{x}\left(\frac{27}{25}\right),\quad \frac{3^{x}}{5^{x}}=\frac{27}{125};\quad \left(\frac{3}{5}\right)^{x}=\left(\frac{3}{5}\right)^{3},$$

wherefrom $x=3$.

**Example 2.11.** Solve the equations:

1) $5^{2x}-23\cdot 5^{x}=50;$ 2) $4^{x}-10\cdot 2^{x-1}-24=0;$

3) $3\cdot\sqrt[x]{81}-10\cdot\sqrt[x]{9}+3=0;$ 4) $9^{x^2-1}-36\cdot 3^{x^2-3}+3=0.$

*Solution.*

1) We denote $2^x=t, (t>0)$. Then the equation $5^{2x}-23\cdot 5^x=50$ converts to the quadratic equation: $t^2-23t-50=0$, wherefrom $t_1=25, t_2=-2<0$, and then $5^x=25; x=2.$

2) We transform the equation and change the variable $2^x=t\ (t>0)$. Then,

$$2^{2x}-10\cdot\frac{2^x}{2}-24=0,\quad 2^{2x}-5\cdot 2^x-24=0,\quad t^2-5t-24=0,$$

$$t_1=8, t_2=-3<0,\quad 2^x=8,\quad x=3.$$

3) We change the variable, $\sqrt[x]{9}=t$, where $t>0$ due to the root definition. Then:

$3t^2-10t+3=0$, $t_1=3, t_2=\frac{1}{3}\notin N$, and from $\sqrt[x]{9}=3$ we obtain $x=2$.

The second root $t_2=\frac{1}{3}$ gives $x=-\frac{1}{2}$, which does not satisfy the original equation. The solution is $x=2$.

4) We transform the equation as follows:

$$3^{2\left(x^2-1\right)}-36\cdot 3^{\left(x^2-1\right)-2}+3=0,$$

$$3^{2\left(x^2-1\right)}-36\cdot\frac{3^{\left(x^2-1\right)}}{3^2}+3=0,$$

$$3^{2\left(x^2-1\right)}-4\cdot 3^{\left(x^2-1\right)}+3=0,$$

and change the variable: $3^{x^2-1}=t\ (t>0)$. Then,

$$t^2-5t-24=0,\quad t_1=1, t_2=3,\quad \text{or}\quad \left[\begin{array}{l} 3^{x^2-1}=1, \\ 3^{x^2-1}=3; \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x^2-1=0, \\ x^2-1=3; \end{array}\right. \Leftrightarrow$$

$$\left[\begin{array}{l} x^2=1, \\ x^2=2; \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x=\pm 1, \\ x=\pm\sqrt{2}; \end{array}\right.$$

i.e., the set $\{-\sqrt{2};-1;1;\sqrt{2}\}$ is the solution.

**Example 2.12.** Solve the equations:

1) $9^x+6^x=2^{2x+1}$; 2) $2\cdot 25^x-5\cdot 10^x+2\cdot 4^x=0$.

*Solution.*

1) After the power transformations, it is obvious that the equation is homogeneous with respect to $2^x$ or $3^x$:

$$3^{2x}+2^x\cdot 3^x=2^{2x}\cdot 2.$$

We divide, for instance, the left and right sides of the equation by $2^{2x}>0$,

$$\left(\frac{3}{2}\right)^{2x}+\left(\frac{3}{2}\right)^{x}-2=0, \text{ and then: } \left(\frac{3}{2}\right)^{x}=t\ (t>0);$$

$$t^2+t-2=0;\quad t_1=1,\quad t_2=-2<0,\quad \left(\frac{3}{2}\right)^{x}=1 \Rightarrow x=0.$$

2) We divide all terms by $4^x$:

$$2\cdot\left(\frac{25}{4}\right)^{x}-5\cdot\left(\frac{10}{4}\right)^{x}+2=0,\ 2\cdot\left(\frac{5}{2}\right)^{2x}-5\cdot\left(\frac{5}{2}\right)^{x}+2=0.$$

Let,

$$\left(\frac{5}{2}\right)^x = t\ (t>0);\quad 2t^2-5t+2=0;\quad t_1=2,\quad t_2=\frac{1}{2}.$$

Then, the solution is:

$$\left[\begin{array}{l}\left(\frac{5}{2}\right)^x = 2,\\ \left(\frac{5}{2}\right)^x = \frac{1}{2};\end{array}\right. \qquad \left[\begin{array}{l} x=\log_{\frac{5}{2}} 2,\\ x=\log_{\frac{5}{2}}\frac{1}{2};\end{array}\right. \qquad \left[\begin{array}{l} x=\log_{\frac{5}{2}} 2,\\ x=\log_{\frac{2}{5}} 2.\end{array}\right.$$

## Tasks for individual work

**Task 2.13.** Solve the equations:

1) $5^{x^2-8x+12}=1$; 2) $2^{x^2}:4^x=8$; 3) $5^{x-4}=6^{x-4}$; 4) $3^{0.5(x-5)}=3\sqrt{3}$;

5) $2^{x^2-135x+1810}=1024$; 6) $10^{x-\sqrt{x^2-5x+1}}=1000$; 7) $3^{x-\sqrt{3x-5}}=27$;

8) $\left(10^{5-x}\right)^{6-x}=100$; 9) $3^{\sqrt{x}+\sqrt[4]{x}-12}=1$; 10) $3^{0.01x^2-0,5x-2,5}=81\sqrt{3}$;

11) $(0.5)^{x^2}2^{2x+3}=64^{-1}$.

**Task 2.14.** Solve the equations:

1) $5^{x+1}+5^x=150$; 2) $2^{x-1}+2^{x-2}+2^{x-3}=448$;

3) $3^{2x-3}-3^{2x-2}+3^{2x}=675$; 4) $5\cdot 2^{\sqrt{x}}-3\cdot 2^{\sqrt{x}-1}=56$;

5) $2\cdot 3^{x-1}-3^{x-2}=5^{x-2}+4\cdot 5^{x-3}$; 6) $5^{2x-1}+2^{2x}-5^{2x}+2^{2x+2}=0$;

7) $7\cdot 3^{x+1}-5^{x+2}=3^{x+4}-5^{x+3}$.

**Task 2.15.** Solve the equations:

1) $9^x-8\cdot 3^x-9=0$; 2) $2\cdot 7^{3x}-5\cdot 49^{3x}+3=0$;

3) $2^{\sqrt{x}}-5\cdot 2^{0.5\sqrt{x}}=24$; 4) $2\cdot 4^{2x}-17\cdot 4^x+8=0$;

5) $5^{1+x^2}-5^{1-x^2}=24$; 6) $5^{2x-1}+5^{x+1}=250$;

7) $2^{2\left(\sqrt{x}-1\right)}-2^{\sqrt{x}}-8=0$; 8) $4^{\sqrt{x-2}}+16=10\cdot 2^{\sqrt{x-2}}$;

9) $2^x+8\cdot 2^{-x}=16.5$.

**Task 2.16.** Solve the equations:

1) $3 \cdot 4^x - 5 \cdot 6^x + 2 \cdot 9^x = 0$; 2) $125 \cdot 25^x - 70 \cdot 10^x + 8 \cdot 4^x = 0$;

3) $3^{2x+4} + 45 \cdot 6^x - 9 \cdot 2^{2x+2} = 0$; 4) $25^x - 10^x = 2^{2x+1}$;

5) $2^{2x+1} - 5 \cdot 6^x + 3^{2x+1} = 0$; 6)* $\left(\sqrt{7+\sqrt{48}}\right)^x + \left(\sqrt{7-\sqrt{48}}\right)^x = 14.$

## 2.6. Logarithmic equations

An equation containing a variable under the logarithm sign is called a logarithmic one. The simplest logarithmic equation of a general form is the following one:

$$\log_a f(x) = \log_a g(x), \left(a > 0, a \neq 1\right), \qquad (2.18)$$

which is equivalent to the system

$$\begin{cases} g(x) > 0, \\ f(x) = g(x). \end{cases}$$

A special case of equation (2.18) is the following:

$$\log_a f(x) = k \quad \Leftrightarrow \quad f(x) = a^k. \qquad (2.19)$$

Let's look at some methods of reducing the logarithmic equations to the simpler ones through examples.

**Example 2.17.** Solve the equations:

1) $\log_{5-x}(2x^2 - 5x + 31) = 2$; 2) $\log_3(x^2 - 4x + 3) = \log_3(3x + 21)$;

3) $\lg(2x) = 2\lg(4x - 15)$; 4) $\log_{x-1}(x^2 + 6) = \log_{x-1}\left(4x^2 - 3x\right)$.

*Solutions.*

1) This equation is equivalent to the system

$$\begin{cases} 2x^2 - 5x + 31 = (5-x)^2, \\ 5 - x > 0, \\ 5 - x \neq 1. \end{cases}$$

A solution of this system: is

$$\begin{cases} x^2+5x+6=(5-x)^2, \\ x<50, \\ x\neq 4. \end{cases} \quad \begin{cases} x_1=-2, \quad x_2=-3; \\ x<5, \\ x\neq 4. \end{cases}$$

wherefrom $x_1=-2, \quad x_2=-3$.

2) Equation (2) is equivalent to the system:

$$\begin{cases} 3x+21>0, \\ x^2-4x+3=3x+21 \end{cases} \Leftrightarrow \begin{cases} x>-7, \\ x^2-7x-18=0 \end{cases} \Leftrightarrow \begin{cases} x>-7 \\ x_1=-2, \quad x_2=9. \end{cases}$$

Both roots, $x_1=-2$ and $x_2=9$, satisfy the inequality $x>-7$. Because of that, these roots are the solutions of equation (2).

3) Equation (3) is equivalent to the system:

$$\begin{cases} 4x-15>0, \\ \lg 2x=\lg(4x-15)^2. \end{cases} \Leftrightarrow \begin{cases} 4x-15>0, \\ 2x=(4x-15)^2. \end{cases}$$

Let us consider the last equation of the second system:

$$2x=16x^2-120x+225 \quad \Leftrightarrow \quad 16x^2-122x+225=0,$$

wherefrom: $x_1=\dfrac{9}{2}, \quad x_2=\dfrac{25}{8}.$ Only $x_1=\dfrac{9}{2}$ satisfies the inequality $4x-15>0$ and therefore it is the solution of equation (3).

4) The equation

$$\log_{x-1}(x^2+6)=\log_{x-1}\left(4x^2-3x\right)$$

is equivalent to the system:

$$\begin{cases} x^2+6>0, \\ x-1>0, \\ x-1\neq 1, \\ x^2+6=4x^2-3x. \end{cases} \Leftrightarrow \begin{cases} x\in R, \\ x-1>0, \\ x-1\neq 1, \\ x^2-x-2=0. \end{cases}$$

Values $x_1=2$ and $x_2=-1$ are the solutions of the system but they do not satisfy the inequalities $x>1$ and $x\neq 2$. Therefore, equation (4) has no solutions.

**Example 2.18.** Solve the equations:

1) $\log_3(x+1)+\log_3(x+3)=1;$ 2) $\lg(x-1)+\lg(x+1)=3\lg 2+\lg(x-2);$

3) $\log_2(x+1)^2+\log_2|x+1|=6;$ 4) $\lg(x-1)^3-3\lg(x-3)=\lg 8.$

*Solution.*

1) Let's find the domain of the logarithmic functions:

$$\begin{cases} x+1>0, \\ x+3>0 \end{cases} \Leftrightarrow \quad x>-1.$$

We remove further the logarithms:

$$\log_3(x+1)+\log_3(x+3)=1; \quad (x+1)(x+3)=3;$$
$$x^2+4x=0; \quad x_1\neq 0, x_2=-4.$$

*Answer:* $x=-4$ does not belong to the domain. Therefore, the solution of equation (1) is $x=-1$.

2) We rewrite the equation taking into account that $3\lg 2=\lg 8$:

$$\lg(x-1)+\lg(x+1)=\lg 8+\lg(x-2).$$

We remove further the logarithms:

$$\begin{cases} \lg(x-1)(x+1) = \lg 8(x-2), \\ x-1>0, \\ x+1>0, \\ x-2>0. \end{cases} \Leftrightarrow \begin{cases} x^2-1=8x-16, \\ x>2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x^2-8x+15=0, \\ x>2 \end{cases} \Leftrightarrow \begin{cases} x_1=3,\ x_2=5, \\ x>2, \end{cases} \Leftrightarrow \left[ \begin{array}{l} x=3, \\ x=5. \end{array} \right.$$

*Answer:* $x=3, x=5.$

3) Consider the equation:

$$\log_2(x+1)^2+\log_2|x+1|=6.$$

The domain of the logarithmic functions is $x+1\neq 0$. That is, $x\neq -1$. Equation (3) is equivalent to the following equation:

$$2\log_2|x+1|+\log_2|x+1|=6 \Rightarrow 3\log_2|x+1|=6,$$

$$\log_2|x+1|=2 \Rightarrow |x+1|=4 \Rightarrow$$

$$\Rightarrow \left[ \begin{array}{l} x+1=4, \\ x+1=-4 \end{array} \right. \Rightarrow \left[ \begin{array}{l} x=3, \\ x=-5. \end{array} \right.$$

*Answer:* $x=3, x=-5.$

4) We transform the equation with the aid of the logarithm rules:

$$3\lg(x-1)-3\lg(x-3)=3\lg 2 \Rightarrow \lg(x-1)=\lg 2+\lg(x-3) \Rightarrow$$

$$\Rightarrow \begin{cases} \lg(x-1)=\lg 2(x-3) \\ x-1>0, \\ x-3>0, \end{cases} \Rightarrow \begin{cases} x-1=2x-6, \\ x>3 \end{cases} \Rightarrow \begin{cases} x=5 \\ x>3 \end{cases} \Rightarrow x=5.$$

*Answer:* $x=5.$

**Example 2.19.** Solve the equations:

1) $\log_2^2 x-3\log_2 x-4=0;$

2) $3\lg x^2-\lg^2(-x)=9;$

3) $\dfrac{1}{5-\lg x}+\dfrac{2}{1+\lg x}=1.$

*Solution.*

1) In this equation $x>0$. We make further the replacement $\log_2 x=t$. Then $t^2-3t-4=0$, wherefrom $t_1=4; t_2=-1$. Let's move on to the variable $x$:

$$\left[\begin{array}{l}\log_2 x=4,\\ \log_2 x=-1.\end{array}\right. \Leftrightarrow \left[\begin{array}{l}x=16,\\ x=\dfrac{1}{2}.\end{array}\right.$$

*Answer*: $x=16; x=0.5.$

2) In this equation $x<0$. Then:

$$6\lg|x|-\lg^2(-x)-9=0,\ 6\lg(-x)-\lg^2(-x)-9=0,$$

$$\lg^2(-x)-6\lg(-x)+9=0 \quad \Rightarrow \quad (\lg(-x)+3)^2=0 \quad \Rightarrow \quad \lg(-x)+3=0$$

$$\lg(-x)=-3 \quad \Rightarrow \quad x=10^{-3}=0.001.$$

Let $\lg x=t$. Then $\dfrac{1}{5-t}+\dfrac{2}{1+t}=1$ or

$$\begin{cases}1+t+10-2t=(5-t)(1+t),\\ t\neq 5,\\ t\neq -1.\end{cases} \Leftrightarrow \begin{cases}t^2-5t+6=0,\\ t\neq 5,\\ t\neq -1.\end{cases} \Leftrightarrow \begin{cases}t_1=2, t_2=3,\\ t\neq 5,\\ t\neq -1.\end{cases}$$

$$\left[\begin{array}{l}t=2,\\ t=3.\end{array}\right. \text{ or } \left[\begin{array}{l}\lg x=2,\\ \lg x=3.\end{array}\right. \Leftrightarrow \left[\begin{array}{l}x=100,\\ x=1000.\end{array}\right.$$

*Answer:* $x=100,\ x=1000.$

**Example 2.20.** Solve the equations:

1) $x^{\lg x}=100x;$ 2) $x^{\log_2 x+2}=8;$ 3) $x^{\frac{\lg x+5}{3}}=10^{\lg x+5}.$

*Solution.*

1) The domain: $x > 0$ and $x \neq 1$. Let us take the common logarithms from both sides:

$$\lg\left(x^{\lg x}\right) = \lg(100x); \quad \lg x \cdot \lg x = \lg 100 + \lg x;$$

$$\lg^2 x - \lg x - 2 = 0.$$

We make further the replacement $\lg x = t$. Then $t^2 - t - 2 = 0$.

The solution of this equation:

$$\left[\begin{array}{l} t = 2, \\ t = -1, \end{array}\right. \Leftrightarrow \left[\begin{array}{l} lg\ x = 2, \\ lg\ x = -1, \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x = 100, \\ x = 0.1. \end{array}\right.$$

2) We take the logarithm to base 2 from both sides of the equation:

$$(\log_2 x + 2)\log_2 x = 3; \quad \log_2^2 x + 2\log_2 x - 3 = 0.$$

We explore further a new variable $\log_2 x = t$. Then $t^2 + 2t - 3 = 0$. The solution of this equation is $t_1 = 1; t_2 = -3$. Then

$$\log_2 x = 1, \quad \text{or} \quad \log_2 x = -3,$$

wherefrom the solution is $x_1 = 2; x_2 = \frac{1}{8}$.

3) We take the logarithm to base 10 from both sides of the equation:

.

$$\frac{\lg x + 5}{3} \lg x = (\lg x + 5)\lg 10,$$

wherefrom $\lg x = -5$ or $\lg x = 3$. Then $x_1 = 10^{-5}$, $x_2 = 1000$.

**Example 2.21.** Solve the equations:

1) $2\log_x 27 - 3\log_{27} x = 1$;

2) $\log_3 x + \log_{\sqrt{3}} x + \log_{\frac{1}{3}} x = 6$;

3) $\log_4(x+12)\cdot\log_x 2=1$;

4) $\lg\sqrt[3]{271+3^{\sqrt{3x}}}=1$.

*Solution.*

1) The domain is $x>0, x\neq 1$. We move on further to base $27$ using formula (2.15):

$$\frac{2}{\log_{27} x}-3\log_{27} x=1$$

and make a replacement, $t=\log_{27} x$. Then we obtain:

$$\frac{2}{t}-3t=1 \quad\Leftrightarrow\quad 2-3t^2=t; \quad 3t^2+t-2=0,$$

wherefrom $t_1=-1, t_2=\frac{2}{3}$. So, $\log_{27} x=-1$ or $\log_{27} x=\frac{2}{3}$, wherefrom the answer is

$$x_1=\frac{1}{27}, x_2=27^{\frac{2}{3}}=\left(3^3\right)^{\frac{2}{3}}=3^2=9.$$

2) The domain is $x>0$. Let us move on to base $3$ using formula (2.12):

$$\log_{\sqrt{3}} x=\log_{3^{\frac{1}{2}}} x=2\log_3 x,$$

$$\log_{\frac{1}{3}} x=\log_{3^{-1}} x=-\log_3 x.$$

So, $\log_3 x+2\log_3 x-\log_3 x=6$, or $\log_3 x=3$. Then the answer is $x=27$

.

3) The domain is:

$$\begin{cases} x+12>0, \\ x>0, \\ x\neq 1, \end{cases} \quad\Rightarrow\quad \begin{cases} x>0, \\ x\neq 1. \end{cases}$$

Let us move on to base 4 and take into account the following equation:

$$\log_x 2 = \log_{x^2} 4 = \frac{1}{\log_4 x^2}.$$

Then

$$\log_4(x+12)\cdot\frac{1}{\log_4 x^2} = 1, \quad \log_4 x^2 = \log_4(x+12), \quad x^2 = x+12,$$

$$x^2 - x - 12 = 0, \quad x_1 = 4,\ x_2 = -3 < 0.$$

*Answer:* $x = 4$.

4) This equation is of the exponentially logarithmic type:

$$\lg\sqrt[3]{271+3^{\sqrt{3x}}} = \lg 10.$$

We have:

$$\sqrt[3]{271+3^{\sqrt{3x}}} = 10,$$

$$271+3^{\sqrt{3x}} = 1000, \quad 3^{\sqrt{3x}} = 729, \quad 3^{\sqrt{3x}} = 3^6 \Rightarrow$$

$$\sqrt{3x} = 6, \quad 3x = 36, x = 12.$$

*Answer:* $x = 12$.

## Tasks for individual work

Solve the equations:

**Task 2.22.** 1) $\log_x\left(x^2-5x+10\right) = 2$; 2) $\log_{x+1}\left(x^2-3x+1\right) = 1$;

3) $\log_{x-3} 7 = 3$; 4) $\log_3\left(x^2-6\right) = \log_3(3x-6)$;

5) $\log_{\frac{1}{3}}\sqrt{2x-1} - \log_{\frac{1}{3}}(x-2) = 0$; 6) $\log_4\log_2\log_{\sqrt{5}} x = \frac{1}{2}$;

7) $\log_3\left(x^2-3\right) - \log_2(3x-5) = 0$; 8) $\log_x\left(|x+1|-2\right) = 2$.

**Task 2.23.** 1) $\frac{\lg(x+33)}{1+\lg 3}=2$; 2) $\lg(x-2)-\frac{1}{2}\lg(3x-6)=\lg 2$;

3) $\lg\sqrt{x+21}+\frac{1}{2}\lg(x-21)=1+\lg 2$; 4) $0.5\lg(2x-1)+\lg\sqrt{x-9}=1$;

5) $\lg 5-1=\lg(x-3)-\frac{1}{2}\lg(3x+1)$.

**Task 2.24.** 1) $4-\lg x=3\sqrt{\lg x}$; 2) $\lg^2 x+\lg x+1=\frac{7}{\lg x-1}$;

3) $2\lg^2(x-1)-10\lg(x-1)+3=0$; 4) $4\lg_4^2(-x)+2\log_4(x^2)=-1$;

5) $\frac{1}{5-4\lg x}+\frac{1}{1+\lg x}=3$; 6) $\log_2^2(-x)+3=2\log_2 x^2$.

**Task 2.25.** 1) $x^{\log_2 x+2}=256$; 2) $0.1\cdot x^{\lg x-2}=100$; 3) $x^{1+\lg x}=100$;

4) $x^{1-\lg x}=0.01$; 5) $x^{\log_3 x-4}=\frac{1}{27}$; 6) $x^{\log_2 x}=4x$.

**Task 2.26.** 1) $\log_5(x+20)\cdot\log_x\sqrt{5}=1$; 2) $\lg\sqrt{75+5\sqrt[3]{x-1}}=1$;

3) $\log_3 x+\log_9 x+\log_{27} x=5.5$; 4) $3^{\log_9(x-7)}=125$;

5) $\log_x 2+\log_2 x=2.5$; 6) $\log_{1-x}3-\log_{1-x}2=\frac{1}{2}$;

7) $\log_2(9^{x-1}+7)=2+\log_2(3^{x-1}+1)$; 8) $6^{\log_6^2 x}+x^{\log_6 x}=12$;

9) $2\cdot\log_x 3\cdot\log_{3x}3=\log_{9\sqrt{x}}3$.

## 2.7. Systems of exponential and logarithmic equations

When solving systems of exponential and logarithmic equations, all known methods for solving systems are used (substitution, algebraic addition of systems of equations, etc.). It is also necessary to remember all the methods for solving the exponential and logarithmic equations discussed earlier.

**Example 2.27.** Solve the system:

$$\begin{cases} 2^x + 2\cdot 3^{x+y} = 56, \\ 3\cdot 2^x + 3^{x+y+1} = 87. \end{cases}$$

*Solution.* Let $2^x = u, 3^{x+y} = v$. Then the system transforms into the following one:

$$\begin{cases} u + 2\cdot v = 56, \\ 3\cdot u + 3\cdot v = 87. \end{cases} \quad \text{or} \quad \begin{cases} u + 2\cdot v = 56, \\ u + v = 29. \end{cases}$$

Subtracting the second equation from the first one, we obtain $u = 27,$ and $v = 2$. Then:

$$\begin{cases} 2^x = 2, \\ 3^{x+y} = 27. \end{cases} \Leftrightarrow \begin{cases} x = 1, \\ x + y = 3. \end{cases} \Leftrightarrow \begin{cases} x = 1, \\ y = 2. \end{cases}$$

**Example 2.28.** Solve the system:

$$\begin{cases} \lg\sqrt{5-x} + \lg 2 = \lg(x+3), \\ x^2 + 7x - 8 = 0. \end{cases}$$

*Solution.* The domain is:

$$\begin{cases} 5 - x > 0, \\ x + 3 > 0; \end{cases} \begin{cases} x < 5, \\ x > -3; \end{cases} \Rightarrow \quad -3 < x < 5.$$

We solve the second equation: $x^2 + 7x - 8 = 0$: $x_1 = 1, x_2 = -8$. The solution $x = -8$ does not belong to the domain and therefore does not fit. We substitute the second solution $x = 1$ in the first equation and obtain the true equality:

$$\lg\sqrt{4} + \lg 2 = \lg 4 \quad \Leftrightarrow \quad 2\lg 2 = 2\lg 2\,.$$

Therefore, the solution of this system is $x = 1.$

**Example 2.29.** Solve the system:

$$\begin{cases} x+y=29, \\ \lg x+\lg y=2\lg 10. \end{cases}$$

*Solution.* The domain is: $x>0, y>0$. Taking into account $\lg 10=1$, we obtain:

$$\begin{cases} x+y=29, \\ \lg(xy)=2. \end{cases} \Leftrightarrow \begin{cases} x+y=29, \\ xy=100. \end{cases} \Leftrightarrow \begin{cases} x=29-y, \\ (29-y)y=100. \end{cases}$$

$$\Leftrightarrow \begin{cases} x=29-y, \\ y^2-29y+100=0. \end{cases} \Leftrightarrow \begin{cases} x=29-y, \\ y_1=4,\ y_2=25. \end{cases} \Leftrightarrow \begin{cases} x=25, \\ y=4. \end{cases} \text{ or } \begin{cases} x=4, \\ y=25. \end{cases}$$

So, the solution is $\{(25;4),(4;25)\}$.

**Example 2.30.** Solve the system:

$$\begin{cases} 3^x \cdot 2^y = 576, \\ \log_{\sqrt{2}}(y-x)=4. \end{cases}$$

*Solution.* The domain is: $y-x>0, y>x$. We can transform the second equation in the following way: $y-x=\left(\sqrt{2}\right)^4$. Then $y-x=4$ and $y=x+4$. Let us substitute further $y=x+4$ into the first equation:

$$3^x \cdot 2^y = 576 \Rightarrow 3^x \cdot 2^{x+4} = 576 \Leftrightarrow 3^x \cdot 2^x \cdot 2^4 = 576 \Rightarrow 6^x = 36 \Rightarrow x=2,$$
$$y=2+4=6.$$

Then, the solution of the system is: $\{(2;6)\}$.

**Example 2.31.** Solve the system:

$$\begin{cases} y-\log_3 x=1, \\ x^y = 3^{12}. \end{cases}$$

*Solution.* The domain is: $x>0, x\neq 1$. Let us apply the base 3 logarithm to both sides of the second equation:

$$\log_3 x^y = \log_3 3^{12} \quad \Rightarrow \quad y\log_3 x = 12.$$

From the first equation we have $\log_3 x = y-1$. Then

$$\begin{cases}\log_3 x = y-1. \\ y^2 - y - 12 = 0.\end{cases} \Rightarrow \begin{cases}\log_3 x = y-1. \\ y_1 = 4;\ y_2 = -3.\end{cases} \Rightarrow \begin{cases}\log_3 x = 3, \\ y = 4.\end{cases}$$

or $\begin{cases}\log_3 x = -4, \\ y = -3.\end{cases} \Rightarrow \begin{cases}x = 27, \\ y = 4.\end{cases} \quad \begin{cases}x = \dfrac{1}{81}, \\ y = -3.\end{cases}$

Therenfore, the solution is $\left\{(27;4);\left(\dfrac{1}{81};-3\right)\right\}$.


$$\begin{cases}\log_3 x = y-1. \\ y^2 - y - 12 = 0.\end{cases} \Rightarrow \begin{cases}\log_3 x = y-1. \\ y_1 = 4;\ y_2 = -3.\end{cases} \Rightarrow \begin{cases}\log_3 x = 3, \\ y = 4.\end{cases}$$ or

$$\begin{cases}\log_3 x = -4, \\ y = -3.\end{cases} \Rightarrow \begin{cases}x = 27, \\ y = 4.\end{cases} \quad \begin{cases}x = \dfrac{1}{81}, \\ y = -3.\end{cases}$$

Therefore, the solution is: $\left\{(27;4);\left(\dfrac{1}{81};-3\right)\right\}$.


## Tasks for individual work

**Task 2.32.** Solve the systems:

1) $\begin{cases}4^{x+y} = 128, \\ 5^{3x-2y-3} = 1;\end{cases}$ 2) $\begin{cases}y - x = 9, \\ \lg y - \lg x = 1;\end{cases}$

3) $\begin{cases} 2^x \cdot 2^y = 6, \\ 3^x \cdot 4^y = 12; \end{cases}$

4) $\begin{cases} 3^x \cdot 2^y = 972, \\ \log_{\sqrt{3}}(x-y) = 2; \end{cases}$

5) $\begin{cases} 4^x \cdot 5^y = 400, \\ 2 \cdot 3^x = 18; \end{cases}$

6) $\begin{cases} 3^{2x} - 2^y = 725, \\ 3^x - 2^{\frac{y}{2}} = 25; \end{cases}$

7) $\begin{cases} 2(\log_y x + \log_x y) = 5, \\ xy = 8; \end{cases}$

8) $\begin{cases} x^y = 5^{12}, \\ \log_5 x + 3^{\log_3 y} = 7; \end{cases}$

9) $\begin{cases} \log_5(x+y) = 1, \\ 2^x + 2^y = 12; \end{cases}$

10) $\begin{cases} \log_4 x + \log_4 y = 1 + \log_4 9, \\ 2^{\frac{x+y}{2}} = 1024. \end{cases}$

## 2.8. Exponential inequalities

Inequalities of both types $a^{f(x)} > a^{g(x)}$ and $a^{f(x)} < a^{g(x)}$, where $a > 0, a \neq 1$, are called the simplest exponential inequalities. Each of these inequalities can be converted into an equivalent inequality depending on the value of the basis $a$:

if $a > 1$, then $a^{f(x)} > a^{g(x)} \Rightarrow f(x) > g(x)$; (2.20)

if $0 < a < 1$, then $a^{f(x)} > a^{g(x)} \Rightarrow f(x) < g(x)$. (2.21)

In these cases, if $a > 1$, the inequality sign is the same as in the initial inequality, but if $0 < a < 1$, the inequality sign is reversed.

**Example 2.33.** Solve the inequality:

$$5^{\frac{2x-1}{x-2}} < 1.$$

*Solution.* So far as $1 = 5^0$, the inequality can be written as

$$5^{\frac{2x-1}{x-2}} < 5^0.$$

So far as the base of the power function is $5>1$, $\frac{2x-1}{x-2}<0$. We solve further the inequality by the interval method (Fig. 2.13.) and obtain the answer:

$$x \in \left(\frac{1}{2};2\right).$$

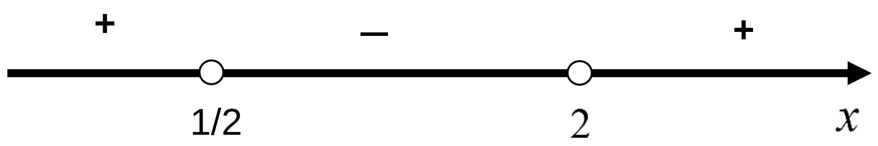



Fig. 2.13. **The solution of the inequality by the interval method**

**Example 2.34.** Solve the inequality:

$$0.3^{4x^2-2x-2} \le 0.3^{2x-3}.$$

*Solution.* So far as the base of the power function is $0.3<1$, then $4x^2-2x-2 \ge 2x-3$ or $4x^2-4x+1 \ge 0$. This inequality holds for all $x \in \Re$.

**Example 2.35.** Solve the inequality:

$$0.2^{\frac{2x-3}{x-2}} \ge 5.$$

*Solution.* The base of the power function is $0.2 = 1/5 = 5^{-1}$. Then the inequality takes the form

$$5^{-\frac{2x-3}{x-2}} \ge 5^1,$$

wherefrom,

$$-\frac{2x-3}{x-2} \ge 1 \text{ or } \frac{2x-3}{x-2} \ge -1.$$

We transform this inequality and solve it by the interval method (Fig. 2.14), i.e.,

$$\frac{2x-3}{x-2}+1\geq 0, \qquad \frac{3x-5}{x-2}\geq 0.$$

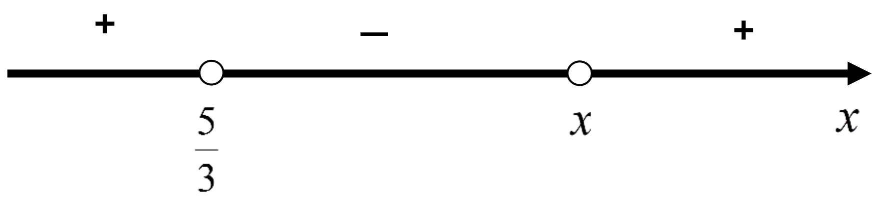



Fig. 2.14. **The solution of the inequality by the interval method**

*Answer:* $x\in\left(-\infty;\frac{5}{3}\right]\cup(2;\infty)$.

**Example 2.36.** Solve the inequality:

$$5^{|4x+2|}<\frac{125}{\sqrt{5^{-6}}}.$$

*Solution.* The right-hand side: $\frac{125}{\sqrt{5^{-6}}}=\frac{125}{5^{-3}}=5^3\cdot 5^3=5^6$. Then,

$$5^{|4x+2|}<5^6 \quad \Leftrightarrow \quad |4x+2|<6 \Leftrightarrow \quad \begin{cases} 4x+2<6 \\ 4x+2>-6 \end{cases} \Leftrightarrow \begin{cases} x<1 \\ x>-2 \end{cases}.$$

*Answer:* $x\in(-2;1)$.

**Example 2.37.** Solve the inequality:

$$4^x-6\cdot 2^x+8<0.$$

*Solution.* Let $2^x=y$, then $4^x=y^2$. We pass further to the system of inequalities:

$$\begin{cases} y^2-6y+8<0, \\ y>0. \end{cases} \Leftrightarrow \begin{cases} 2<y<4, \\ y>0. \end{cases} \Leftrightarrow \quad y<4.$$

We return to the variable $x$ and obtain $2<2^x<4$, wherefrom $1<x<2$.

*Answer:* $x \in (1;2)$.

**Example 2.38.** Solve the inequality:

$$3\cdot 5^x-5^{2x+1}+2<0.$$

*Solution.* We transform the inequality and make the replacement

$$5^x=y, y>0.$$

Then

$$3\cdot 5^x-5^{2x}\cdot 5+2<0, \qquad 5^{2x}\cdot 5-3\cdot 5^x-2>0, \text{ or:}$$

$$\begin{cases} 5y^2-3y-2>0, \\ y>0. \end{cases}$$

We solve this system of inequalities by the interval method (Fig. 2.15). The solution of the quadratic equation $5y^2-3y-2=0$ is

$$y_1=1, y_2=-\frac{2}{5}.$$

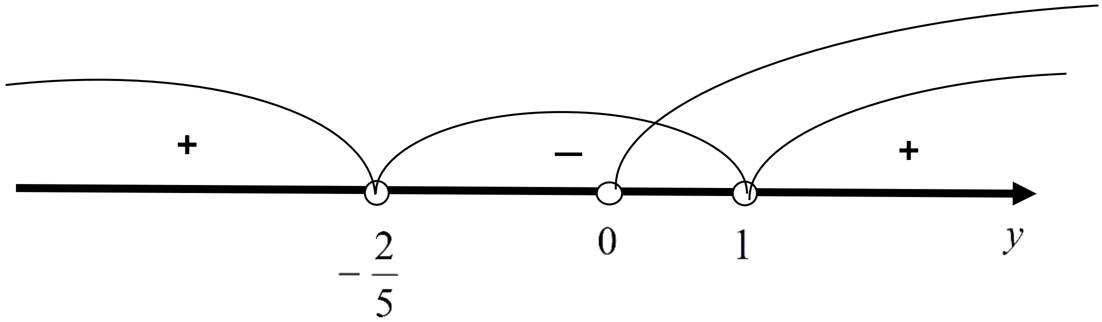



Fig. 2.15. **The solution of the inequality by the interval method**

The solution for $y$ is $y>1$, i.e., $5^x>1$ or $5^x>5^0$, wherefrom $x>0$, i.e., the solution for $x$ is $x\in(0;\infty)$.

## Tasks for individual work

Solve these exponential inequalities.

**Task 2.39.** 1) $2^{4x}<16$; 2) $\left(\frac{1}{0.125}\right)^x\le 128$; 3) $2^{3-2x-x^2}>1$;

4) $(0.3)^{2x^2-x+3}<0.00243$; 5) $\left(\frac{1}{5}\right)^{\frac{2x+1}{1-x}}>\left(\frac{1}{5}\right)^{-3}$; 6) $\left(\frac{2}{5}\right)^{\frac{6-5x}{2+5x}}<\frac{25}{4}$;

7) $\frac{2^{x^2}}{4^x}<8$; 8) $2^{x^2-6x+0.5}\le\left(16\sqrt{2}\right)^{-1}$; 9) $5^{x+1}<\left(\frac{1}{25}\right)^{\frac{1}{x}}$;

10) $3^{|3x-4|}\le 9^{2x-3}$; 11) $5^{|4x-6|}\ge 25^{3x-4}$; 12) $2^{\sqrt{x^2-3x+3}}>2^{\sqrt{x}}$.

**Example 2.40.** 1) $3^{x+2}-3^x-72>0$; 2) $2^x-2^{x-4}-15<0$;
3) $5^{x-3}\ge 7^{3-x}$; 4) $11^{x-7}<17^{7-x}$; 5) $2^{5x-1}+2^{5x-2}+2^{5x-3}<896$;
6) $5\cdot 2^{\sqrt{x}}-3\cdot 2^{\sqrt{x}-1}\le 56$; 7) $3\cdot 2^{x+1}+5\cdot 2^x-2^{x+2}\le 21$.

## 2.9. Logarithmic inequalities

The simplest logarithmic inequalities have the following form:

$\log_a f(x)>\log_a g(x)$ or $\log_a f(x)<\log_a g(x)$ , if $a>0, a\ne 1$.

Each of these inequalities, depending on the base sign, can be transformed into an equivalent system of inequalities:

for $a>1$: $$\log_a f(x)>\log_a g(x) \iff \begin{cases} f(x)>g(x), \\ f(x)>0. \end{cases} \qquad (2.22)$$

for $0<a<1$ $\quad \log_a f(x) > \log_a g(x) \quad \Leftrightarrow \quad \begin{cases} f(x) < g(x), \\ f(x) > 0. \end{cases}$ (2.23)

**Example 2.41.** Solve the inequalities:

1) $\log_5(1-x) < \log_5(x+3)$; 2) $\log_{0.5}(x^2+1) \le \log_{0.5}(2x-5)$.

*Solution.*

1) This inequality is equivalent to the system of inequalities:

$$\begin{cases} 1-x>0, \\ x+3>0, \\ 1-x<x+3. \end{cases} \Leftrightarrow \begin{cases} x<1, \\ x>-3, \\ x>-1. \end{cases} \Leftrightarrow -1<x<1,$$

therefore the solution is: $x \in (-1;1)$.

2)The inequality is equivalent to the system of inequalities, given that the base of the logarithm is less than one, $0.5<1$:

$$\begin{cases} x^2+1>0, \\ 2x-5>0, \\ x^2+1 \ge 2x-5. \end{cases} \Leftrightarrow \begin{cases} x \in R, \\ x>2.5>0, \\ x^2-2x+6 \ge 0. \end{cases} \Leftrightarrow \begin{cases} x \in R, \\ x>2.5, \\ x \in R, \text{ because } D<0. \end{cases}$$

*Answer:* $x \in (2.5;\infty)$.

**Example 2.42.** Solve the inequalities:

1) $\log_3(3x-1) < 1$; 2) $\log_{\frac{1}{6}}(x^2-3x+2) < -1$;

3) $\log_{\frac{1}{4}} \dfrac{x-3}{x+3} \ge -\dfrac{1}{2}$; 4) $\log_{0.3} \log_6 \dfrac{x^2+x}{x+4} < 0$.

Solutions.

1) The inequality $\log_3(3x-1) < \log_3 3$ is equivalent to the system:

$$\begin{cases} 3x-1>0, \\ 3x-1<3. \end{cases} \quad \begin{cases} x>\frac{1}{3}, \\ x<\frac{4}{3}. \end{cases} \Leftrightarrow \quad \frac{1}{3}<x<\frac{4}{3}.$$

Wherefrom the answer is $x\in\left(\frac{1}{3};\frac{4}{3}\right)$.

2) We transform the inequality as follows:

$$\log_{\frac{1}{6}}(x^2-3x+2)<-1 \quad \Leftrightarrow \quad \log_{\frac{1}{6}}(x^2-3x+2)<\log_{\frac{1}{6}}\frac{1}{6} \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \begin{cases} x^2-3x+2>0, \\ x^2-3x+2>6. \end{cases} \quad \Leftrightarrow \quad x^2-3x+2>6 \quad \Leftrightarrow \quad x^2-3x-4>0.$$

The roots of $x^2-3x-4=0$ are $x_1=-1,\quad x_2=4$. The graphical solution of the inequality is presented in Fig. 2.16.

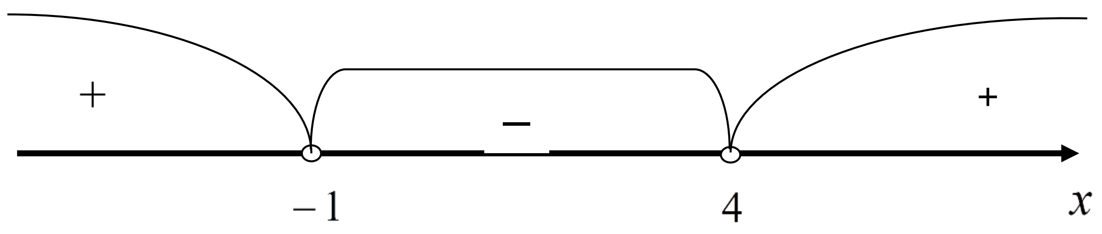



Fig. 2.16. **The graphical solution to the inequality**

*Answer:* $x\in(-\infty;-1)\cup(4;\infty)$.

3) The inequality

$$\log_{\frac{1}{4}}\frac{x-3}{x+3}\geq-\frac{1}{2}$$

is equivalent to the following one:

$$\log_{\frac{1}{4}} \frac{x-3}{x+3} \geq \log_{\frac{1}{4}} \left(\frac{1}{4}\right)^{-\frac{1}{2}},$$

which, in turn, is equivalent to the following system of inequalities:

$$\begin{cases} \dfrac{x-3}{x+3} > 0, \\ \dfrac{x-3}{x+3} \leq 2. \end{cases} \Leftrightarrow \begin{cases} \dfrac{x-3}{x+3} > 0, \\ \dfrac{x-3}{x+3} - 2 \leq 0. \end{cases} \Leftrightarrow \begin{cases} \dfrac{x-3}{x+3} > 0, \\ \dfrac{-x-9}{x+3} \leq 0. \end{cases} \Leftrightarrow \begin{cases} \dfrac{x-3}{x+3} > 0, \\ \dfrac{x+9}{x+3} \geq 0. \end{cases}$$

We solve each inequality by the interval method (Fig. 2.17, 2.18), and then find the intersection of the obtained solutions (Fig. 2.19):

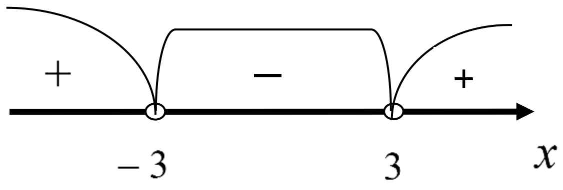



Fig. 2.17. **The solution of the 1st inequality**

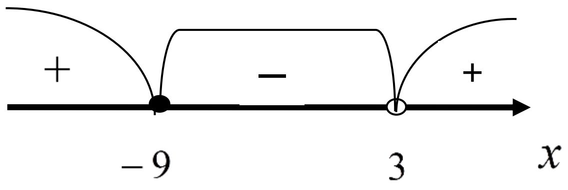



Fig. 2.18. **The solution of the 2nd inequality**

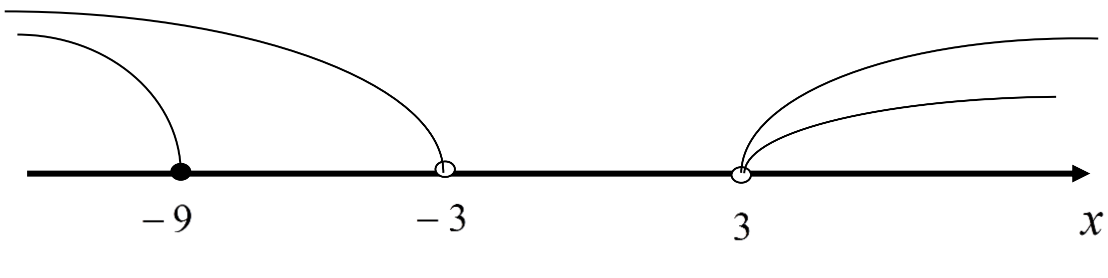



Fig. 2.19. **The common solution**

4) Transform the inequality

$$\log_{0.3} \log_6 \frac{x^2 + x}{x+4} < 0$$

into the equivalent system:

$$\log_{0.3}\log_6\frac{x^2+x}{x+4}<\log_{0.3}1 \quad\Leftrightarrow\quad \begin{cases}\log_6\dfrac{x^2+x}{x+4}>1,\\ \log_6\dfrac{x^2+x}{x+4}>0\end{cases} \quad\Leftrightarrow$$

$$\log_6\frac{x^2+x}{x+4}>\log_6 6. \quad\Leftrightarrow\quad \begin{cases}\dfrac{x^2+x}{x+4}>6,\\ \dfrac{x^2+x}{x+4}>0.\end{cases} \quad\Leftrightarrow\quad \frac{x^2+x}{x+4}>6.$$

We solve further the last inequality:

$$\frac{x^2+x}{x+4}-6>0, \frac{x^2+x-6x-24}{x+4}>0, \frac{x^2-5x-24}{x+4}>0, \frac{(x-8)(x+3)}{x+4}>0.$$

We solve the inequality using the interval method (Fig. 2.20):

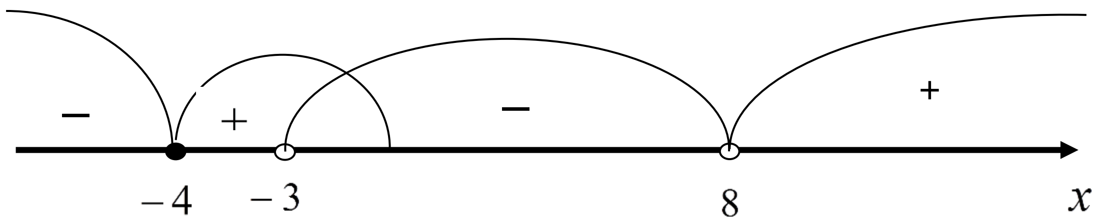



Fig. 2.20. **The solution of the inequality by the interval method**

*Answer:* $x\in(-4;-3)\cup(8;\infty)$.

**Example 2.43.** Solve the inequalities:

$$1)\ \frac{\log_2(x+1)}{x-1}>0;\quad 2)\ (x+1)\log_4(x+2)<0.$$

*Solution.* For each of these inequalities we apply the interval method taking into account their domain.

1) The domains is:

$$\begin{cases} x+1>0, \\ x-1\neq 0. \end{cases} \quad \begin{cases} x>-1, \\ x\neq 1. \end{cases} \quad \text{or } x\in(-1;1)\cup(1;\infty).$$

On the number axis for $x>-1$, we take the points in which the numerator and the denominator are zero. The solution is presented in Fig. 2.21.

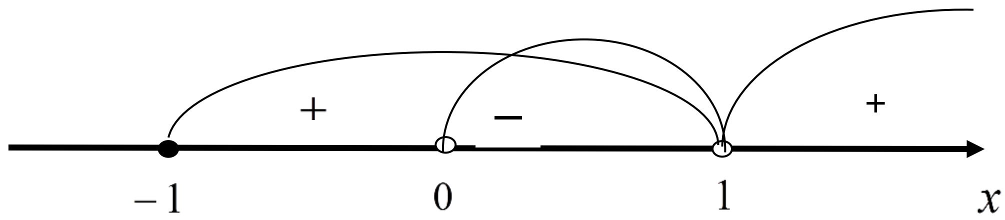



Fig. 2.21. **The solution of the inequality by the interval method**

2) Here, the domain is:

$$x>-4, \text{ i.e. } x\in(-1;0)\cup(1;\infty).$$

For the interval method, we take the points $x=-1$ and $x=-3$, because $\log_4(x+4)=0$ for $x=-3$ (Fig. 2.22).

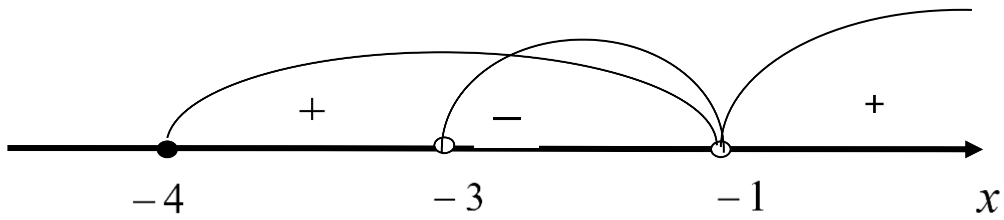



Fig. 2.22. **The solution of the inequality by the interval method**

*Answer:* $x\in(-3;-1)$.

**Example 2.44.** Solve the inequalities:

1) $\log_3|x-1|<1;$ 2) $|4-\log_2 x|<2;$ 3) $\log_2(|x-2|-1)<1;$

*Solutions.*

1) The domain is:

$$\begin{cases} |x-1| < 3, \\ x \neq 1, \end{cases}$$

Using this domain, we have:

$$\begin{cases} x-1 < 3, \\ x-1 > -3, \\ x \neq 1, \end{cases}$$

wherefrom we obtain the answer: $x \in (-2;1) \cup (1;4)$.

2) We convert the inequality into a more convenient form:

$$|\log_2 x - 4| < 2,$$

using the module property: $|a| = |-a|$. Next we pass from the inequality to the system of inequalities using the theorem:

$$|f(x)| < a \Leftrightarrow \begin{cases} f(x) < a, \\ f(x) > -a. \end{cases}$$

Then,

$$|\log_2 x - 4| < 2 \Leftrightarrow \begin{cases} \log_2 x - 4 < 2, \\ \log_2 x - 4 > -2. \end{cases} \Leftrightarrow \begin{cases} \log_2 x < 6, \\ \log_2 x > 2. \end{cases} \Leftrightarrow \begin{cases} 0 < x < 2^6, \\ x > 4. \end{cases} \Leftrightarrow$$

$$\Leftrightarrow 4 < x < 64, \text{ i.e. } x \in (4; 64).$$

*Answer:* $x \in (4;64)$.

3) Because $1 = \log_2 2$, the inequality takes the form:

$$\log_2(|x-2| - 1) < \log_2 2.$$

We pass on to a system of inequalities that allows us to solve the problem graphically by the interval method (Fig. 2.23):

$$\begin{cases}|x-2|-1>0,\\ |x-2|-1<2,\end{cases}\Leftrightarrow\begin{cases}|x-2|>1,\\ |x-2|<3,\end{cases}\Leftrightarrow\begin{cases}\left[\begin{array}{l}x-2>1,\\ x-2<-1,\end{array}\right.\\ \begin{cases}x-2<3,\\ x-2>-3,\end{cases}\end{cases}\Leftrightarrow\begin{cases}\left[\begin{array}{l}x>3,\\ x<1,\end{array}\right.\\ \begin{cases}x<5,\\ x>-1,\end{cases}\end{cases}\Leftrightarrow\begin{cases}\left[\begin{array}{l}x>3,\\ x<1,\end{array}\right.\\ -1<x<5.\end{cases}$$

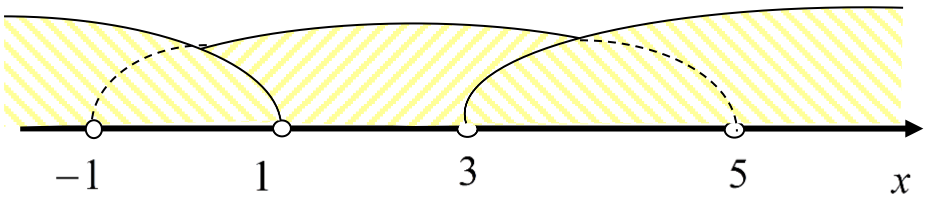



Fig. 2.23. **The graphical solution**

As can be seen from the figure, the answer is $x\in(-2;1)\cup(3;5)$.

**Example 2.45.** Solve the inequalities:

$$1)\ \log_{x-1}(x+1)>2;\quad 2)\ \log_{\frac{1}{x}}\left(|2.5x-1|-1\right)>-2.$$

*Solution.*

1) We consider two cases:

a) If $x-1>1, x>2$, we have the system:

$$\begin{cases}x>2,\\ x+1>(x-1)^2.\end{cases}\Leftrightarrow\begin{cases}x>2,\\ x^2-3x<0.\end{cases}\Leftrightarrow\begin{cases}x>2,\\ x(x-3)<0.\end{cases}\Leftrightarrow\begin{cases}x>2,\\ 0<x<3.\end{cases}.$$

The answer is: $2<x<3.$

b) If $0<x-1<1, 1<x<2$, we have:

$$\begin{cases}1<x<2,\\ 0<x+1>(x-1)^2.\end{cases}\Leftrightarrow\begin{cases}1<x<2,\\ x^2-3x>0.\end{cases}\Leftrightarrow\begin{cases}1<x<2,\\ \left[\begin{array}{l}x<0,\\ x>3.\end{array}\right.\end{cases}$$

The graphical solution of the case b) by the interval method is shown in Fig. 2.24.

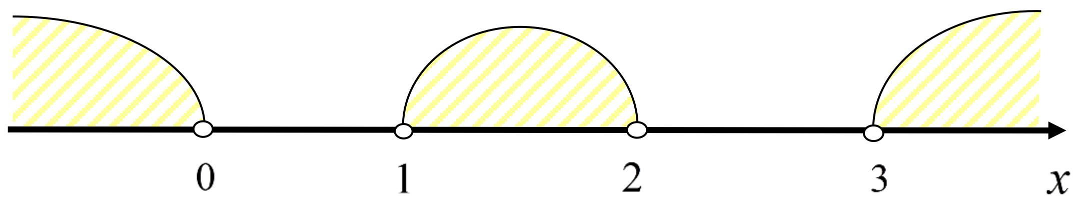



Fig. 2.24. **The graphical solution**

This system of inequalities has no solutions. Therefore, the solution obtained in the case a) remains.

*Answer:* $2 < x < 3$.

2) The domain here is $2.5x - 1 > 0$. We consider two cases:

$$\text{a) } \begin{cases} \frac{1}{x} > 1, \\ 2.5x - 1 > 0, \\ 2.5x - 1 > \left(\frac{1}{x}\right)^{-2}. \end{cases} \Leftrightarrow \begin{cases} \frac{1-x}{x} > 0, \\ x > \frac{2}{5}, \\ x^2 - 2.5x + 1 < 0. \end{cases} \Leftrightarrow \begin{cases} 1 - x > 0, \\ x > \frac{2}{5}, \\ x^2 - 2.5x + 1 < 0. \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{2}{5} < x < 1, \\ x^2 - 2.5x + 1 < 0. \end{cases} \Leftrightarrow \begin{cases} \frac{2}{5} < x < 1, \\ \frac{1}{2} < x < 2. \end{cases} \Leftrightarrow \frac{1}{2} < x < 2.$$

The roots of the equation $x^2 - 2.5x + 1 = 0$ : $x_1 = \frac{1}{2}$, $x_2 = 2$. So, the solution of the inequality $x^2 - 2.5x + 1 < 0$ is the interval: $\frac{1}{2} < x < 2$.

b) In the second case we have $0 < \frac{1}{x} < 1$. Then,

$$\begin{cases} \frac{1}{x} - 1 < 0, \\ x > 0. \end{cases} \Leftrightarrow \begin{cases} \frac{1-x}{x} < 0, \\ x > 0. \end{cases} \Leftrightarrow \begin{cases} 1 - x < 0, \\ x > 0. \end{cases} \Leftrightarrow \begin{cases} x > 1, \\ x > 0. \end{cases} \Leftrightarrow x > 1.$$

The original inequality is equivalent to the system of inequalities:

$$\begin{cases} 2.5x-1>0, \\ 0<\dfrac{1}{x}<1, \\ 2.5x-1<\left(\dfrac{1}{x}\right)^{-2}. \end{cases} \Leftrightarrow \begin{cases} x>\dfrac{2}{5}, \\ x>1, \\ x^2-2.5x+1>0. \end{cases} \Leftrightarrow \begin{cases} x>1, \\ \left[\begin{array}{l} x>2, \\ x<\dfrac{1}{2}. \end{array}\right. \end{cases}$$

The solution of the last inequality is given in Fig. 2.25.

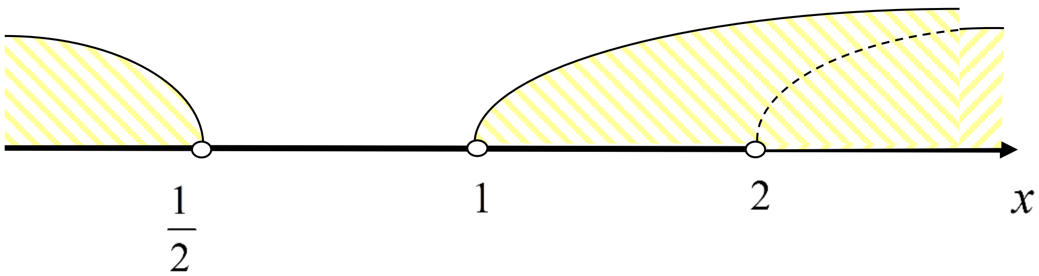



Fig. 2.25. **The graphical solution by the interval method**

The answer for the second case is: $x>2$. Combining the results, we obtain a common solution: $x\in\left(\dfrac{1}{2},1\right)\cup(2;\infty)$.

**Example 2. 46.** Solve the inequality:

$$\left(\frac{2}{5}\right)^{\log_{0.25}\left(x^2-5x+8\right)}\le 2.5.$$

*Solution.* This is an exponential inequality. Because

$$2.5=\frac{5}{2}=\left(\frac{2}{5}\right)^{-1},$$

we have:

$$\left(\frac{2}{5}\right)^{\log_{0.25}\left(x^2-5x+8\right)} \le 2.5,$$

wherefrom:

$$\log_{0.25}\left(x^2-5x+8\right) \ge -1 \Leftrightarrow \begin{cases} x^2-5x+8>0 \\ x^2-5x+8 \le (0.25)^{-1} \end{cases}.$$

The first inequality holds for any $x \in R$ because the discriminant of the equation $x^2-5x+8=0$ is negative. The equation $x^2-5x+4=0$ has the following roots: $x_1=1, x_2=4$. Therefore, the inequality $x^2-5x+4 \le 0$ holds in the interval $1 \le x \le 4$. Because of that, a common solution of the original system is $x \in [1;4]$.

## Tasks for individual work

Solve the logarithmic inequality:

**Task 2.47.** 1) $\log_3(1-2x) \ge \log_3(5x-2)$;

2) $\log_7(2-x) \ge \log_7(3x+6)$;

3) $\log_{\frac{1}{3}}(x+4) < \log_{\frac{1}{3}}\left(x^2+2x-2\right)$; 4) $\log_{\frac{1}{5}}(3x-1) \ge \log_{\frac{1}{5}}(3-x)$;

5) $\log_{\frac{1}{11}}(2x+21) < -2$; 6) $\log_{\frac{1}{2}}\left(x^2-5x+6\right) \ge -1$;

7) $\lg\left(x^2-5x+7\right) < 0$; 8) $\log_{1.5}\frac{2x-8}{x-2} < 0$;

9) $\log_2\frac{x}{x-1} \le -1$; 10) $\log_{0.1}\log_2\frac{x^2+1}{x-1} \le 0$.

**Task 2.48.** 1) $\frac{\log_2(x-1)}{x-3} \le 0$; 2) $(x-3)\log_{\frac{1}{7}}(x+8) \ge 0$;

3) $\dfrac{x-5}{\log_3(x-2)} \geq 0$;  4) $(x-6)\log_5(x-3) > 0$;

5) $\dfrac{\log_3(x-3)}{x^2-25} > 0$;  6) $\dfrac{\log_{0.3}|x-2|}{x^2-4x} < 0.$

**Task 2.49.** 1) $\log_{3x-2} x \leq 1$;  2) $\log_x \dfrac{4x-2}{3} \geq 1$;

3) $\log_x |x-2| < 1$.

**Task 2.50.** 1) $\log_5(26-3^x) > 2$;  2) $5^{\log_3 \frac{2}{x+2}} \leq 1$;

3) $\left(\dfrac{1}{2}\right)^{\log_2\left(x^2-1\right)} > 1$;  4) $\lg^2 x - \lg x - 2 \geq 0.$

**Task 2.51.** Find the domain of the functions:

1) $y = \log_2\left(\left(x^2-3x\right)(x+5)\right)$;  2) $y = \dfrac{\lg\left(3-2x-x^2\right)}{\sqrt{x}}$;

3) $y = \sqrt{\log_3 \dfrac{3-x}{x}}$;  4) $y = \sqrt{4x-x^2} + \lg\left(x^2-1\right)$;

5) $y = \sqrt{\log_{0.3}\left(x^2-5x+7\right)}$;  6) $y = \dfrac{\sqrt{x^2-2x}}{\log_2(x-1)}$;  7) $y = \sqrt{\lg \dfrac{1-2x}{x+3}}$.

## Questions for self-assessment

1. What is a logarithm?
2. What is the base of a logarithm?
3. What is the exponent of a logarithm?
4. What is the logarithm $\lg b$ called?
5. Name the main logarithmic identity.
6. Name the basic logarithm laws.
7. What is a logarithmic function?
8. What is the difference between the graphs of the logarithmic function for $a > 1$ and the logarithmic function for $0 < a < 1$?
9. What is a logarithmic equation?

10. What is a general form of the simplest logarithmic equation?
11. Name all the methods of solving the logarithmic equations.
12. What is an exponential function?
13. Name the main properties of an exponential function.
14. What is the difference between the graphs of the exponential function for $a > 1$ and the logarithmic function for $0 < a < 1$?
15. What is an exponential equation?
16. Name all the methods of solving the exponential equations.
17. What is the system of exponential and logarithmic equations?
18. What are the simplest exponential inequalities?
19. What are the simplest logarithmic inequalities?
20. What is the interval method of solving an inequality?

# 3. Limits and derivatives of functions

## 3.1. The limit of a function

The number $A$ is called the limit of the function $f(x)$ as $x$ approaches $x_0 \left(x \to x_0\right)$, if, for any arbitrary small $\varepsilon > 0$, there exists the number $\delta > 0$, such that from inequality $|x - x_0| < \delta$ the inequality $|f(x) - A| < \varepsilon$ follows. This is denoted as $\lim\limits_{x \to x_0} f(x) = A$.

The function $y = f(x)$ is called the infinitesimal function if it tends to zero if $x \to x_0$:

$$\lim_{x \to x_0} f(x) = 0.$$

The function $y = f(x)$ is called unlimited when $x \to x_0$, if for any $M > 0$ there exists such a number $\delta > 0$, that from the inequality $|x - x_0| < \delta$, the inequality $|f(x)| > M$ follows. The limit of such a function equals $\pm\infty$,

$$\lim_{x \to x_0} f(x) = \pm\infty.$$

The evaluation of limits is based on the following rules (if there exist the limits $\lim\limits_{x\to x_0} f(x)$ and $\lim\limits_{x\to x_0} g(x)$):

1) $\lim\limits_{x\to x_0}\left(f(x)+g(x)\right)=\lim\limits_{x\to x_0} f(x)+\lim\limits_{x\to x_0} g(x)$;

2) $\lim\limits_{x\to x_0}\left(f(x)\cdot g(x)\right)=\lim\limits_{x\to x_0} f(x)\cdot\lim\limits_{x\to x_0} g(x)$;

$\lim\limits_{x\to x_0}\left(a\cdot f(x)\right)=a\cdot\lim\limits_{x\to x_0} f(x)$, $a=const$;

3) $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)}$, if $\lim\limits_{x\to x_0} g(x)\neq 0$.

The following properties are also used ($c>0$):

$\lim\limits_{x\to\infty} cx=\infty$; $\lim\limits_{x\to\infty}\dfrac{x}{c}=\infty$;

$\lim\limits_{x\to\infty}\dfrac{c}{x}=0$; $\lim\limits_{x\to+0}\dfrac{c}{x}=+\infty$;

$\lim\limits_{x\to-0}\dfrac{c}{x}=-\infty$.

In the simplest cases, calculating a limit reduces to substituting the boundary value of the argument into the given function. Often, however, such a substitution leads to uncertainties of the following forms:

$$\left[\frac{0}{0}\right];\quad \left[\frac{\infty}{\infty}\right];\quad \left[0\cdot\infty\right];\quad \left[\infty-\infty\right];\quad \left[1^{\infty}\right];\quad \left[0^{0}\right];\quad \left[\infty^{0}\right].$$

Finding a limit in these cases is called *the solution of uncertainties*. To solve an uncertainty, you must first convert this function and then pass on to the limit.

The following examples show the methods for solving uncertainties.

**Example 3.1.** Evaluate the limit:

$$\lim_{x\to 2}\left(3x^2-2x+7\right).$$

*Solution.* Using the limit theorems, we have:

$$\lim_{x\to 2}\left(3x^2-2x+7\right)=3\lim_{x\to 2}x\cdot\lim_{x\to 2}x-2\lim_{x\to 2}x+\lim_{x\to 2}7=$$
$$=3\cdot 2\cdot 2-2\cdot 2+7=15.$$

**Example 3.2.** Evaluate the limit:

$$\lim_{x\to 3}\frac{3x^2-1}{2x^3+6x^2-5}.$$

*Solution.* The limits of the numerator and denominator exist and the denominator limit is not equal to zero. Using the fraction limit theorem, we have:

$$\lim_{x\to 3}\frac{3x^2-1}{2x^3+6x^2-5}=\frac{\lim_{x\to 3}\left(3x^2-1\right)}{\lim_{x\to 3}\left(2x^3+6x^2-5\right)}=\frac{3\cdot 9-1}{2\cdot 27+6\cdot 9-5}=\frac{26}{103}.$$

**Example 3.3.** Evaluate the limit:

$$\lim_{x\to 2}\frac{x-2}{x^2+3x-1}.$$

*Solution.* The limit of the numerator and denominator exists and the denominator limit is not equal to zero. Using the fraction limit theorem, we have:

$$\lim_{x\to 2}\frac{x-2}{x^2+3x-1}=\frac{0}{9}=0.$$

**Example 3.4.** Evaluate the limit:

$$\lim_{x \to 4} \frac{x^2 - 6x + 8}{x^2 - 5x + 4}.$$

*Solution.* If $x \to 4$, the numerator and denominator are approaching zero, i.e., we have the uncertainty of the type $\left[\frac{0}{0}\right]$. To solve this uncertainty, we factor the numerator and denominator and reduce the fraction:

$$\lim_{x \to 4} \frac{x^2 - 6x + 8}{x^2 - 5x + 4} = \lim_{x \to 4} \frac{(x-4)(x-2)}{(x-4)(x-1)} = \lim_{x \to 4} \frac{x-2}{x-1} = \frac{2}{3}.$$

**Example 3.5.** Evaluate the limit:

$$\lim_{x \to 1} \frac{3x^3 + 2x^2 - x - 4}{2x^3 - 5x + 3}.$$

*Solution.* In this example, we also have the uncertainty $\left[\frac{0}{0}\right]$. To solve this uncertainty, we factor the numerator and denominator. The first factor in both the numerator and denominator will be $x - 1$, and the other one is found by dividing both polynomials by $x - 1$:

$$\begin{array}{r|l} 3x^3 + 2x^2 - x - 4 & x - 1 \\ \underline{3x^3 - 3x^2} \qquad\qquad & \overline{3x^2 + 5x + 4} \\ 5x^2 - x \qquad & \\ \underline{5x^2 - 5x} \qquad & \\ 4x - 4 & \\ \underline{4x - 4} & \\ 0 & \end{array} \qquad \begin{array}{r|l} 2x^3 \qquad - 5x + 3 & x - 1 \\ \underline{2x^3 - 2x^2} \qquad\qquad & \overline{2x^2 + 2x - 3} \\ 2x^2 - 5x \qquad & \\ \underline{2x^2 - 2x} \qquad & \\ -3x + 3 & \\ \underline{-3x + 3} & \\ 0 & \end{array}.$$

Then we have:

$$\lim_{x\to 1}\frac{(x-1)\left(3x^2+5x+4\right)}{(x-1)\left(2x^2+2x-3\right)}=\lim_{x\to 1}\frac{3x^2+5x+4}{2x^2+2x-3}=\frac{12}{1}=12.$$

**Example 3.6.** Evaluate the limit:

$$\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}.$$

*Solution.* In this example, we also have the uncertainty $\left[\frac{0}{0}\right]$.

To solve an uncertainty, when there is irrationality, we transfer it from the numerator to the denominator or vice versa. After that, the resulting fraction can be simplified and transformed to a limit.

In this example, we multiply the numerator and denominator by the multiplier conjugated to the numerator, and then, after simplification of the fraction, we get:

$$\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}=\lim_{x\to 0}\frac{\left(\sqrt{x+4}-2\right)\left(\sqrt{x+4}+2\right)}{x\left(\sqrt{x+4}+2\right)}=$$

$$=\lim_{x\to 0}\frac{x+4-4}{x\left(\sqrt{x+4}+2\right)}=\lim_{x\to 0}\frac{1}{\sqrt{x+4}+2}=\frac{1}{4}.$$

**Example 3.7.** Evaluate the limit:

$$\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}.$$

*Solution.* In this example, we also have the uncertainty $\left[\frac{0}{0}\right]$. We multiply the numerator and denominator by the product

$$\left(\sqrt{x^2+1}+1\right)\left(\sqrt{x^2+16}+4\right).$$

Then we obtain:

$$\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}=$$

$$=\lim_{x\to 0}\frac{\left(\sqrt{x^2+1}+1\right)\left(\sqrt{x^2+1}+1\right)\left(\sqrt{x^2+16}+4\right)}{\left(\sqrt{x^2+16}-4\right)\left(\sqrt{x^2+1}+1\right)\left(\sqrt{x^2+16}+4\right)}=$$

$$=\lim_{x\to 0}\frac{\left(x^2+1-1\right)\left(\sqrt{x^2+16}+4\right)}{\left(x^2+16-16\right)\left(\sqrt{x^2+1}+1\right)}=$$

$$=\lim_{x\to 0}\frac{\sqrt{x^2+16}+4}{\sqrt{x^2+1}+1}=\frac{8}{2}=4.$$

**Example 3.8.** Evaluate the limit:

$$\lim_{x\to\infty}\frac{2x^3+x^2-1}{3x^3-x^2+x-3}.$$

*Solution.* In this example, we have the uncertainty $\left[\frac{\infty}{\infty}\right]$*.* In such examples, the numerator and denominator need to be divided by $x^n$, where $n$ is the largest of the powers of the numerator and denominator. Then we can pass on to a limit. Namely, dividing the numerator and denominator by $x^3$, we have :

$$\lim_{x\to\infty}\frac{2x^3+x^2-1}{3x^3-x^2+x-3}=\lim_{x\to\infty}\frac{2+\frac{1}{x}-\frac{1}{x^3}}{3-\frac{1}{x}+\frac{1}{x^2}-\frac{3}{x^3}}=\frac{2}{3},$$

because

$$x \to \infty \Rightarrow \frac{1}{x} \to 0,\ \frac{1}{x^2} \to 0,\ \frac{1}{x^3} \to 0.$$

**Example 3.9.** Evaluate the limit:

$$\lim_{x\to\infty} \frac{x^4 + 3x - 1}{x^2 - 5}.$$

*Solution.* In this example, we also have the uncertainty $\left[\frac{\infty}{\infty}\right]$. Dividing the numerator and denominator by $x^4$, we have:

$$\lim_{x\to\infty} \frac{x^4 + 3x - 1}{x^2 - 5} = \lim_{x\to\infty} \frac{1 + \frac{3}{x^3} - \frac{1}{x^4}}{\frac{1}{x^2} - \frac{5}{x^4}} = \infty.$$

**Example 3.10.** Evaluate the limit:

$$\lim_{x\to\infty} \frac{x^2 + 3x - 1}{x^3 + 2}.$$

*Solution.* In this example, we have the uncertainty $\left[\frac{\infty}{\infty}\right]$. Dividing the numerator and denominator by $x^3$, we have:

$$\lim_{x\to\infty} \frac{x^2 + 3x - 1}{x^3 + 2} = \lim_{x\to\infty} \frac{\frac{1}{x} + \frac{3}{x^2} - \frac{1}{x^3}}{1 + \frac{2}{x^3}} = \frac{0}{1} = 0.$$

**Example 3.11.** Evaluate the limit:

$$\lim_{x\to 3}\left(\frac{1}{x-3}-\frac{6}{x^2-9}\right).$$

*Solution.* In this example, we have the uncertainty $[\infty-\infty]$. We bring the fractions to a common denominator:

$$\lim_{x\to 3}\left(\frac{1}{x-3}-\frac{6}{x^2-9}\right)=\lim_{x\to 3}\frac{x-3}{x^2-9}=\lim_{x\to 3}\frac{x-3}{(x-3)(x+3)}=$$

$$=\lim_{x\to 3}\frac{1}{x+3}=\frac{1}{6}.$$

## Tasks for individual work

Find the limits:

**Task 3.12.** $\lim_{x\to 0}\frac{x^3-x^2+5}{x^2-2x}$.

**Task 3.13.** $\lim_{x\to 3}\frac{x^2-9}{x-3}$.

**Task 3.14.** $\lim_{x\to 2}\frac{x-2}{x^3-8}$.

**Task 3.15.** $\lim_{x\to 1}\frac{x^2+2x-3}{x^2-1}$.

**Task 3.16.** $\lim_{x\to -1}\frac{x^2-4x-5}{x^2-2x-3}$.

**Task 3.17.** $\lim_{x\to 3}\frac{3x^2-11x+6}{2x^2-5x-3}$.

**Task 3.18.** $\lim_{x\to 1}\frac{x^2+x-2}{x^3-x^2-x+1}$.

**Task 3.19.** $\lim_{x\to 0}\frac{x}{\sqrt{x+4}-2}$.

**Task 3.20.** $\lim_{x\to 3}\frac{\sqrt{2x+3}-3}{3-x}$.

**Task 3.21.** $\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$.

**Task 3.22.** $\lim_{x\to \infty}\frac{x^2+5x-4}{2x^2-4x+3}$.

**Task 3.23.** $\lim_{x\to \infty}\frac{3x^3-2x+4}{x^2+3x-1}$.

**Task 3.24.** $\lim_{x\to \infty}\frac{2x^2+5x-4}{x^3-4x^2+2}$.

**Task 3.25.** $\lim_{x\to \infty}\left(x-\frac{x^3}{x^2+1}\right)$.

**Task 3.26.** $\lim_{x\to \infty}\left(\frac{x^2}{x+3}-x\right)$.

## 3.2. The differentiation

Let the function $y = f(x)$ be defined in any point $x$ and in some neighborhood of this point. The derivative of this function at the point $x$ is called the limit:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

The derivatives are denoted by the characters:

$$y', \quad y'(x), \quad \frac{dy}{dx}, \quad f'(x).$$

### The differentiation rules

1. $C' = 0 \quad (C = const)$; 2. $(Cu)' = C \cdot u'$;

3. $(u \pm v)' = u' \pm v'$; 4. $(uv)' = u'v + uv'$;

5. $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$;

6. $y' = f'(u) \cdot u'$, if $y = f(u)$, and $u = u(x)$, i.e., $y$ is a composite function of $x$.

### The table of derivatives

1. $(x^n)' = nx^{n-1}$, 2. $(x)' = 1$,

3. $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$, 4. $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$,

5. $(\sin x)' = \cos x$, 6. $(\cos x)' = -\sin x$,

7. $(\mathrm{tg} x)' = \frac{1}{\cos^2 x}$, 8. $(\mathrm{ctg} x)' = -\frac{1}{\sin^2 x}$

9. $(a^x)' = a^x \cdot \ln a$, 10. $(e^x)' = e^x$,

11. $(\log_a x)' = \frac{1}{x \ln a}$, 12. $(\ln x)' = \frac{1}{x}$,

13. $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$,  14. $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$,

15. $(\mathrm{arctg}x)' = \dfrac{1}{1+x^2}$,  16. $(\mathrm{arcctg}x)' = -\dfrac{1}{1+x^2}$.

**Example 3.27.** Find the derivative of $y = \frac{1}{5}x^5 + 2\sqrt{x}$.

*Solution.* Using the rules (2 and 3), as well as the table of derivatives (formula 1), we have:

$$y' = \left(\frac{1}{5}x^5 + 2\sqrt{x}\right)' = \left(\frac{1}{5}x^5\right)' + \left(2\sqrt{x}\right)' = \frac{1}{5}\left(x^5\right)' + 2\left(x^{\frac{1}{2}}\right)' =$$

$$= \frac{1}{5}\cdot 5x^4 + 2\cdot\frac{1}{2}x^{-\frac{1}{2}} = x^4 + \frac{1}{\sqrt{x}}.$$

**Example 3.28.** Find the derivative of $y = x \cdot \cos x$.

*Solution.* Using rule 4 as well as the table of derivatives (formulas 2 and 6), we have:

$$y' = (x\cdot\cos x)' = x'\cos x + x(\cos x)' = \cos x - x\sin x.$$

**Example 3.29.** Find the derivative of $y = \dfrac{\arcsin x}{x^2}$.

*Solution.* Using rule 5 as well as the table of derivatives (formulas 1 and 13), we have:

$$y' = \left(\frac{\arcsin x}{x^2}\right)' = \frac{(\arcsin x)' x^2 - \left(x^2\right)' \arcsin x}{x^4} =$$

$$= \frac{x^2 - 2x\sqrt{1-x^2}\arcsin x}{x^4\sqrt{1-x^2}} = \frac{x - 2\sqrt{1-x^2}\arcsin x}{x^3\sqrt{1-x^2}}.$$

**Example 3.30.** Find the derivative of

$$y = \sin(2x+3).$$

*Solution.* Using the rules (6 and 2), as well as the table of derivatives (formulas 2 and 5), we have:

$$y' = (\sin(2x+3))' = \cos(2x+3)\cdot(2x+3)' = 2\cos(2x+3).$$

**Example 3.31.** Find the derivative of

$$y = \mathrm{tg}\ln x.$$

*Solution.* Using rule 6 as well as the table of derivatives (formulas 7 and 12), we have:

$$y' = (\mathrm{tg}\ln x)' = \frac{1}{\cos^2 \ln x}(\ln x)' = \frac{1}{\cos^2 \ln x}\frac{1}{x}.$$

**Example 3.32.** Find the derivative of

$$y = x^2 \sin^2 x^2.$$

*Solution.* Using the rules (4 and 6), as well as the table of derivatives (formulas 1 and 5), we have:

$$y' = \left(x^2 \sin^2 x^2\right)' = \left(x^2\right)' \cdot x^2 \sin^2 x^2 + x^2\left(\sin^2 x^2\right)' =$$

$$= 2x \cdot x^2 \sin^2 x^2 + 2\sin x^2 \cdot x^2\left(\sin x^2\right)' = 2\cdot x^3 \sin^2 x^2 +$$

$$+ 2\sin x^2 \cdot x^2\left(\cos x^2\right)\cdot 2x = 2x\left(\sin^2 x^2 + x^2 \sin 2x^2\right).$$

## Tasks for individual work

Find the derivatives:

**Task 3.33.** $y = 5x^2 - 2x + 5e^x$.

**Task 3.34.** $y = 2x^{\frac{7}{2}} + \sqrt{x^3}$.

**Task 3.35.** $y = (3x-2)(7x+5)$.

**Task 3.36.** $y = (4x^2 + x)(x^3 - 2x)$.

**Task 3.37.** $y = \dfrac{5x^2 - 3}{3 - 4x}$.

**Task 3.38.** $y = (3x^2 + 4)^5$.

**Task 3.39.** $y = \left(5x^3 - \dfrac{1}{x}\right)^4$.

**Task 3.40.** $y = \sqrt{1 + 2x^2}$.

**Task 3.41.** $y = \sqrt[3]{8x^2 - 1}$.

**Task 3.42.** $y = \dfrac{x}{1 - \cos x}$.

**Task 3.43.** $y = 3\sin x^2$.

**Task 3.44.** $y = 2\sin^3 x$.

**Task 3.45.** $y = 3\cos(3x - 5)$.

**Task 3.46.** $y = \operatorname{tg} \dfrac{1}{x}$.

**Task 3.47.** $y = \operatorname{ctg}^2 x$.

**Task 3.48.** $y = (\arcsin x)^2$.

**Task 3.49.** $y = \ln(2x+1)$.

**Task 3.50.** $y = \ln \sin x$.

**Task 3.51.** $y = \ln^3(x+2)$.

**Task 3.52.** $y = 10^{2x-1}$.

**Task 3.53.** $y = e^{\sqrt{x+1}}$.

**Task 3.54.** $y = 3^{\sin 2x}$.

**Task 3.55.** $y = e^{\sqrt{\ln x}}$.

**Task 3.56.** $y = e^{\arcsin 2x}$.

## 3.3. The geometric meaning of the derivative

The derivative of the function $y = f(x)$ at the point $x = x_0$ is the angular coefficient of the tangent drawn to the graph of this function at the point with the abscissa $x_0$, i.e.,

$$y'(x_0) = \operatorname{tg}\alpha .$$

The equation of the tangent to the graph of this function at the point $M(x_0; y_0)$ has the form:

$$y - y_0 = y'(x_0)(x - x_0).$$

**Example 3.57.** Find the tangent equation to the curve $y = x^2 - 1$ at the point $M(1;0)$.

*Solution.* The equation of the tangent to the graph of this function at the point $M(x_0; y_0)$ has the form:

$$y - y_0 = y'(x_0)(x - x_0).$$

We find the angular coefficient of the tangent drawn to the graph of this function $y = x^2 - 1$ at the point with the abscissa $x_0 = 1$:

$$f'(x) = 2x.$$

From this, $f'(1) = 2$. So, the tangent equation has the form:

$$y - 0 = 2(x - 1), \text{ or } 2x - y - 2 = 0.$$

**Example 3.58.** Find the tangent equation to the curve $y = \frac{3}{x}$ at the point $M(1;3)$.

*Solution.* We find the slope of the tangent to the graph of this function at $M(1;3)$:

$$f'(x) = -\frac{3}{x^2}.$$

From this, we have $f'(1) = -3$. So, the tangent equation has the form:

$$y - 3 = -3(x - 1), \text{ or } 3x + y - 6 = 0.$$

**Example 3.59.** Find the tangent equation to the curve $y = \frac{x+1}{x-1}$ at the point with the abscissa $x_0 = 2$.

*Solution.* Let us find $y_0 = y(0)$:

$$y_0 = \frac{2+1}{2-1} = 3.$$

Then the tangent point has the coordinates $M(2;3)$. We further find the slope

$$y' = \frac{(x-1)-(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

and then

$$y'(x_0) = y'(2) = \frac{-2}{(2-1)^2} = -2.$$

The tangent equation has the form:

$$y-3 = -2(x-2) \text{ or } y = -2x+7.$$

## Tasks for individual work

**Task 3.60.** Find the tangent equation to the curve $y = x^2 + 4x - 3$ at the point $M(1;2)$.

**Task 3.61.** Find the tangent equation to the curve $y = x^3 + 4x^2 - 1$ at the point with the abscissa $x_0 = -1$.

**Task 3.62.** Find the tangent equation to the curve $y = \arccos 3x$ at the point of this function graph intersection with the axis $Oy$.

## 3.4. The investigation of functions

A function $f(x)$ is called increasing on the interval $(a,b)$ if for any two points $x_1$ and $x_2$ from this interval the following statement holds: $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$ (Fig. 3.1).

A function $f(x)$ is called decreasing on the interval $(a,b)$ if for any two points $x_1$ and $x_2$ from this interval the following statement holds: $x_2 > x_1$ $\Rightarrow f(x_2) < f(x_1)$ (Fig. 3.2).

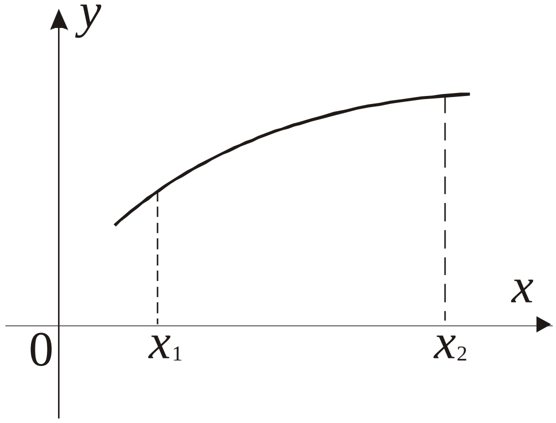


Fig. 3.1. **An increasing function**

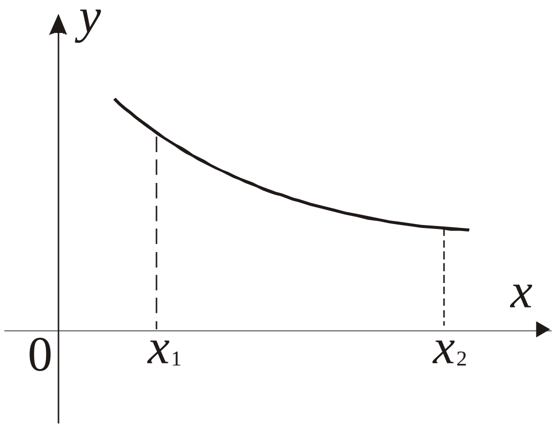


Fig. 3.2. **A decreasing function**

The intervals of increasing and decreasing functions are called the intervals of *monotonicity* of a function. To determine them, the following attribute is used: if the derivative $y' > 0$ at all points of the interval $(a,b)$, the function increases on this interval, and if $y' < 0$, it decreases.

A point $x_0$ at which $f'(x_0)$ is equal to zero or does not exist is called *critical*. If, when passing through a critical point, the derivative changes its sign from "+" to "-", then at this point there is a *maximum* (*max*); if the derivative changes its sign from "-" to "+", then at this point there is a *minimum* (*min*).

The maximum and minimum points are called *extremum* points. Only those points at which the derivative is zero or does not exist can serve as extremum points.

**Example 3.63.** Find the intervals of monotonicity and the extremum points of the function:

$$y = \frac{x^3}{3} - x^2 - 3x.$$

*Solution.* The domain is $x \in R$, because of that this function is a polynomial. We find the derivative:

$$y' = \frac{3x^2}{3} - 2x - 3 = x^2 - 2x - 3.$$

We further find the critical points from the equation $y' = 0$:

$$x^2 - 2x - 3 = 0.$$

We study the derivative graphically (Fig. 3.3).

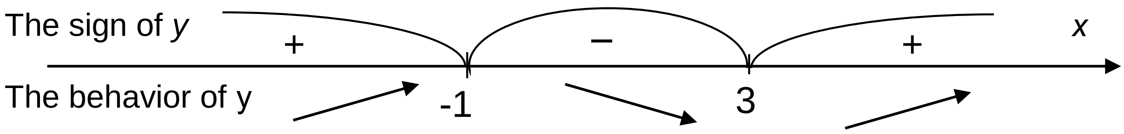



Fig. 3.3. **The graphical study of the derivative**

The critical points are $x_1 = 3$, $x_2 = -1$. We put these points on the numerical axis. They break it into three intervals. Find out the sign of the derivative in each of the obtained intervals. It is recommended that the derivative be presented as a product: $y' = (x-3)(x+1)$.

As follows from Fig. 3.3, the function has a maximum at $x = -1$, and a minimum at $x = 3$, moreover:

$$y_{\max} = \frac{-1}{3} - 1 + 3 = \frac{5}{3}, \text{ and } y_{\min} = \frac{3^3}{3} - 9 - 9 = -9.$$

On the intervals $(-\infty; -1)$ and $(3; \infty)$ the function increases, but on the interval $(-1;\ 3)$ the function decreases.

**Example 3.64.** Investigate the monotonicity and the extremum of the function $y = x - \ln(1+x)$.

*Solution.* The domain of this function is determined from the solution of the inequality $1 + x > 0$, wherefrom $x > -1$. Let us find $y'$:

$$y' = 1 - \frac{1}{1+x} = \frac{x}{x+1}.$$

Solving the equation $\frac{x}{x+1} = 0$, we find the critical point $x = 0$ $(x > -1)$

that belongs to the domain of this function. We examine this point on an extremum graphically (Fig. 3.4).

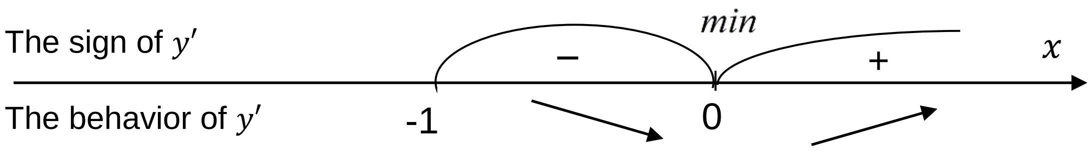



Fig. 3.4. **The graphical study of the derivative**

As we can see from Fig. 3.4, the function decreases with $x \in (-1;0)$ and increases with $x \in (0;\infty)$. At $x = 0$ the function has a minimum: $y_{\min} = y(0) = \ln 1 = 0$.

## The global extrema of a function

If a function $y = f(x)$ is continuous within the segment $[a;b]$, then it reaches its largest and smallest value in this segment. These values are achieved either at the ends of the segment, or at the extremum points.

To determine the extremum points, one needs to find all the critical points belonging to the segment, then calculate the values of the function at these points and at the ends of the segment, and then choose the largest and smallest ones from all obtained values. They are denoted as follows: the largest – $\max\limits_{[a,b]} f(x)$, the smallest – $\min\limits_{[a,b]} f(x)$.

**Example 3.65.** Find the largest and smallest values of the function $f(x) = 2x^3 - 3x^2 - 12x + 10$ on the segment $[-2, 1]$.

*Solution.* Let's find the derivative $f'(x) = 6x^2 - 6x - 12$, and the critical points from the equation $6x^2 - 6x - 12 = 0$: $x_1 = -1$, $x_2 = 2$. Only $x = -1$ belongs to the segment $[-2, 1]$.

$$f(-1) = 17; \qquad f(-2) = 6; \qquad f(1) = -3.$$

Let us calculate the values of the function at this point and at the ends of this segment. We get:

$$f(-1) = 17; \qquad f(-2) = 6; \qquad f(1) = -3.$$

From the obtained values, we select the largest and smallest ones:

$$\max_{[-2,1]} f(x)=17, \qquad \min_{[-2,1]} f(x)=-3.$$

## The general scheme of investigation of a function and plotting a graph

The investigation of a function should be done according to the following steps:

find the domain;

investigate the function for evenness and oddness;

find the points of intersection with coordinate axes (not obvious);

find the intervals of monotonicity and the extremal points;

plot the function graph using the obtained results.

**Example 3.66.** Investigate the function $y=\frac{1}{3}x^3-\frac{3}{2}x^2-4x+10$ and plot its graph.

*Solution.*

1. The domain is $x \in R$, because this function is a polynomial.

2. The function is neither even or odd (of the general form), the graph does not have a symmetry.

3. The points of intersection with coordinate axes are:

a) with $Oy$: $\quad x=0,\ y=10;$

b) with $Ox$: $\quad y=0,\ \frac{1}{3}x^3-\frac{3}{2}x^2-4x+10=0.$

The solution to this equation is difficult. Therefore, we find the points of intersection with the axis $Ox$ approximately when plotting a graph.

4. The intervals of increasing and decreasing of the function and the extremal points are found from the function derivative:

$$y'=\frac{1}{3}\cdot 3x^2-\frac{3}{2}\cdot 2x-4=x^2-3x-4.$$

We can find the critical points, solving the equation $x^2 - 3x - 4 = 0$, wherefrom we have: $x_1 = -1$, $x_2 = 4$, and then $y' = (x+1)(x-4)$.

Then we plot the critical points on the numerical axis (Fig. 3.5) and get three intervals. Let us find out the sign of the derivative and the behavior of the function on each of the intervals graphically:

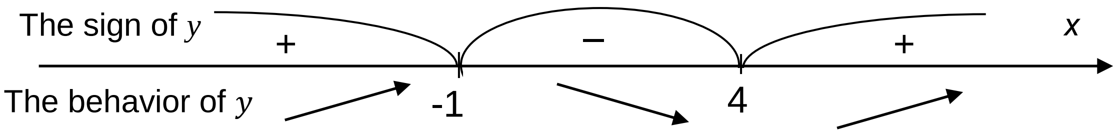



Fig. 3.5. **Investigation of the function** $y = \frac{1}{3}x^3 - \frac{3}{2}x^2 - 4x + 10$

The extrema:

$$y_{\max}(-1) = -\frac{1}{3} - \frac{3}{2} + 4 + 10 = 12\frac{1}{6}.$$

$$y_{\min}(4) = \frac{1}{3}\cdot 64 - \frac{3}{2}\cdot 16 - 4\cdot 4 + 10 = -8\frac{2}{3}.$$

Based on the data obtained, it is also possible to plot the function graph.

## Tasks for individual work

Find the intervals of monotonicity and the extrema of the functions:

**Task 3.67.** $y = 2x^3 - 6x^2 - 18x + 7$. **Task 3.68.** $y = 2x^3 - 3x^2$.

**Task 3.69.** $y = 2x^2 - \ln x$. **Task 3.70.** $y = x - \ln(1 + x^2)$.

Find the largest and smallest values of the functions in the indicated intervals:

**Task 3.71.** $y = x^3 - 3x^2 + 1$, $x \in [-1; 4]$.

**Task 3.72.** $y = x^3 + 3x^2 - 9x - 7$, $x \in [-4; 3]$.

**Task 3.73.** $y = x \ln^2 x$, $x \in \left[\frac{1}{e}; e\right]$.

**Task 3.74.** $y = \ln x - 2\operatorname{arctg} x, \qquad x \in [1;\infty)$.

Solve the problems:

**Task 3.75.** The difference between two numbers is 13. What should these numbers be for their product to be the smallest?

**Task 3.76.** Into the semicircle of radius $R$, inscribe a rectangle with the largest perimeter.

Conduct a complete investigation of the functions and plot their graphs:

**Task 3.77** $y = x^3 - 4x^2 - 3x + 6$. **Task 3.78** $y = 6x^2 - 9x - x^3$.

**Task 3.79** $y = \dfrac{x^2}{x-3}$. **Task 3.80** $y = x^2 - \dfrac{8}{x}$.

**Task 3.81** $y = \ln\left(x^2 + 4\right)$.

## Questions for self-assessment

1. What is the limit of the function $f(x)$ as $x$ approaches $x_0$?
2. What is an infinitesimal function?
3. Name the rules for limits.
4. Name the limit theorems.
5. Name all forms of uncertainties.
6. What is the solution of uncertainties called?
7. What is the derivative of the function $f(x)$ at the point $x$?
8. Name the differentiation rules.
9. Name the table of derivatives.
10. What is the geometric meaning of the derivative?
11. What is the angular coefficient of the tangent?
12. Name the equation of a tangent to the graph of the function $f(x)$ at the point $M(x_0, y_0)$.
13. What is an increasing function $f(x)$ on the interval $(a,b)$ called?
14. What is a decreasing function $f(x)$ on the interval $(a,b)$ called?
15. What are the intervals of monotonicity of a function?
16. What is a maximum of a function?
17. What is a minimum of a function?

18. What is the smallest $\left(\min\limits_{[a,b]} f(x)\right)$ value of a function on a segment?

19. What is the largest $\left(\max\limits_{[a,b]} f(x)\right)$ value of a function on a segment?

20. What are the critical points of a function?

# 4. The antiderivative functions and the integral

## 4.1. The antiderivative

A function $F(x)$ is called an antiderivative of the function $f(x)$ on a given interval if for all $x$ from this interval the following equality holds:

$$F'(x) = f(x). \tag{4.1}$$

For instance, the function $F_1(x) = 3x - 2\cos x$ is an antiderivative of the function $f_1(x) = 3 + 2\sin x$ for all $x \in R$, and the function $F_2(x) = -5\ln x$ is an antiderivative of the function $f_2(x) = \frac{-5}{x}$ for all $x > 0$. As we can see, the primitives of the functions $f_1(x)$ and $f_2(x)$ are also the functions:

$$F_1(x) = 3x - 2\cos x + c \text{ and } F_2(x) = 3 - 5\ln x + c,$$

where $c$ is an arbitrary real number (constant). Really,

$$(3x - 2\cos x)' = (3x - 2\cos x + c)' = 3 + 2\sin x,$$

$$(-5\ln x)' = (-5\ln x + c)' = -\frac{5}{x}.$$

So, if $F(x)$ is a primitive of the function $f(x)$, then $F(x) + c$ is also a primitive of the same function $f(x)$.

The set of all antiderivatives of a function $f(x)$ is called the indefinite integral of the function $f(x)$ and denoted as

$$\int f(x)dx = F(x) + c.$$

To find the primitive $F(x)$, the three basic rules are commonly applied:

1. If $F_1(x)$ is the primitive of $f_1(x)$ and $F_2(x)$ is the primitive of $f_2(x)$, the function $F_1(x) + F_2(x)$ is the primitive of $f_1(x) + f_2(x)$.

2. If $F(x)$ is the primitive of $f(x)$, the function $mF(x)$ is the primitive of $m\,f(x)$.

3. If $F(x)$ is the primitive of $f(x)$, the function $\frac{1}{a}F(ax+b)$ is the primitive of $f(ax+b)$.

The finding of antiderivatives for given functions is called the integration (Table 4.1).

Table 4.1

**The table of antiderivatives**

| Function $f(x)$ | Antiderivative $F(x)$ |
|---|---|
| $k$ | $kx + C$ |
| $x^n$ | $\frac{x^{n+1}}{n+1} + C, n \neq -1$ |
| $\frac{1}{x}$ | $\ln\lvert x\rvert + C$ |
| $\sin x$ | $\cos x + C$ |
| $\cos x$ | $-\sin x + C$ |
| $\frac{1}{\cos^2 x}$ | $\mathrm{tg}x + C$ |
| $\frac{1}{\sin^2 x}$ | $\mathrm{ctg}x + C$ |
| $a^x$ | $\frac{a^x}{\ln a} + C$ |
| $e^x$ | $e^x + C$ |

**Example 4.1.** Prove that the function $F(x)=(x+1)\sqrt[3]{x^2}+C$ is a primitive of the function $f(x)=\dfrac{5x+2}{\sqrt[3]{x}}+C$ for all $x\in(0;\infty)$.

*Solution.* Really, we can find $F'(x)$ using the rule $(uv)'=u'v+v'u$:

$$F'(x)=\sqrt[3]{x^2}+(x+1)\frac{2}{3\sqrt[3]{x}}=\frac{3x+2x+2}{3\sqrt[3]{x}}=\frac{5x+2}{3\sqrt[3]{x}}.$$

So, $F'(x)=f(x)$ for all $x\in(0;\infty)$, i.e., the function $F(x)=(x+1)\sqrt[3]{x^2}$ is really a primitive of the function $f(x)=\dfrac{5x+2}{3\sqrt[3]{x}}$ .

**Example 4.2.** Find the indefinite integral of the function $f(x)=(\sin x+\cos x)^2$ whose graph passes through the point $M(0;0.5)$.

*Solution.* We transform the given function in such a way that we can apply the table of antiderivatives and the integration rules:

$$f(x)=(\sin x+\cos x)^2=\sin^2 x+2\sin x\cdot\cos x+\cos^2 x=1+\sin 2x.$$

Then, the integral:

$$F(x)=x-\frac{\cos 2x}{2}+c.$$

We find an arbitrary constant given that $F(0)=0.5$:

$\dfrac{1}{2}=0-\dfrac{\cos 0}{2}+c$, wherefrom $c=1$. So, $F(x)=1+x-\dfrac{\cos 2x}{2}$ .

*Answer:* $F(x)=1+x-\dfrac{\cos 2x}{2}$.

**Example 4.3.** Find the indefinite integrals of the functions:

1) $f(x)=(3x+7)^{10}$;

2) $f(x)=\sqrt{5-2x}$;

3) $f(x)=\frac{1}{x}+\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}$; 4) $f(x)=\cos(1-2x)+3x^2$;

5) $f(x)=\frac{2}{3x+1}$; 6) $f(x)=e^{-2x}+e^{2x}$;

7) $f(x)=\sin 3x\cdot\sin 2x$; 8) $f(x)=\sin^2 x$.

*Solution.* Each of the specified functions must be converted to a table form and the integration rules must be applied. Then,

1) for $f(x)=(3x+7)^{10}$, $F(x)=\frac{1}{3}\frac{(3x+7)^{11}}{11}+C=\frac{(3x+7)^{11}}{33}+C$;

2) for $f(x)=\sqrt{5-2x}$, $F(x)=-\frac{\sqrt{(5-2x)^3}}{3}+C$,

3) for $f(x)=\frac{1}{x}+\frac{1}{(x+1)^2}+\frac{1}{(x+2)^3}=\frac{1}{x}+(x+1)^{-2}+(x+2)^{-3}$,

$$F(x)=\ln|x|+\frac{(x+1)^{-1}}{-1}+\frac{(x+1)^{-2}}{-2}=\ln|x|-\frac{1}{x+1}-\frac{1}{2(x+2)^2}+c;$$

4) for $f(x)=\cos(1-2x)+3x^2$,

$$F(x)=-\frac{\sin(1-2x)}{-2}+x^3+c=\frac{\sin(1-2x)}{2}+x^3+c;$$

5) for $f(x)=\frac{2}{3x+1}$, $F(x)=2\frac{\ln(3x+1)}{3}+c=\frac{2}{3}\ln(3x+1)+c$;

6) for $f(x)=e^{-2x}+e^{2x}$,

$$F(x)=-\frac{1}{2}e^{-2x}+\frac{1}{2}e^{2x}+c=\frac{e^{-2x}-e^{2x}}{2}+c;$$

7) for $f(x)=\sin 3x\cdot\sin 2x=\frac{1}{2}(\cos x-\cos 5x)$,

$$F(x)=\frac{1}{2}\left(\sin x-\frac{\sin 5x}{5}\right)+c=\frac{\sin x}{2}-\frac{\sin 5x}{10}+c;$$

8) for $f(x)=\sin^2 x$, $F(x)=\frac{1}{2}x-\frac{\sin 2x}{2}+c=\frac{x}{2}-\frac{\sin 2x}{4}+c$.

## 4.2. The definite integral

If $f(x)$ is continuous on the interval $[a;b]$ and $F(x)$ is any antiderivative of $f(x)$, i.e. $F'(x)=f(x)$, then the number $F(b)-F(a)$ is called the definite integral of $f(x)$ with lower ($a$) and upper ($b$) limits and evaluated as

$$\int_a^b f(x)dx=F(b)-F(a). \tag{4.2}$$

This one is called the first fundamental theorem of calculus. It is convenient to use it in the following form:

$$\int_a^b f(x)dx=F(x)\Big|_a^b=F(b)-F(a). \tag{4.3}$$

Recall the simplest properties of the definite integrals:

$$\int_a^b \big(f_1(x)+f_2(x)\big)dx=\int_a^b f_1(x)dx+\int_a^b f_2(x)dx; \tag{4.4}$$

$$\int_a^b mf(x)dx=m\int_a^b f(x)dx; \tag{4.5}$$

$$\int_a^b f(cx+d)dx=\frac{1}{c}F(cx+d)\Big|_a^b. \tag{4.6}$$

**Example 4.4.** Calculate the integral:

$$\int_{-1}^{2}\left(3x^2-4x+3\right)dx.$$

*Solution.*

$$\int_{-1}^{2}\left(3x^2-4x+3\right)dx=3\cdot\frac{x^2}{3}-4\cdot\frac{x^2}{2}+3x\Bigg|_{-1}^{2}=$$

$$=\left(x^3-2\cdot x^2+3x\right)\Big|_{-1}^{2}=(8-8+6)-(-1-2-3)=12.$$

**Example 4.5.** Calculate the integral:

$$\int_{1}^{e^2}\frac{2\sqrt{x}+5-7x}{x}dx.$$

*Solution.* We transform the integrand into a sum and find the antiderivative for this sum. Then:

$$\int_{1}^{e^2}\frac{2\sqrt{x}+5-7x}{x}dx=\int_{1}^{e^2}\left(2x^{-\frac{1}{2}}+\frac{5}{x}-7\right)dx=$$

$$=2\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+5\ln|x|-7x\Bigg|_{1}^{e^2}=\left(4\sqrt{x}+5\ln x-7x\right)\Big|_{1}^{e^2}=$$

$$=\left(4\sqrt{e^2}+5\ln e^2-7e^2\right)-(4+5\ln 1-7)=$$

$$=4e+10\ln e-7e^2+3=4e-7e^2+13.$$

**Example 4.6.** Calculate the integral:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos^2 x dx.$$

*Solution.*

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos^2 x dx=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1+\cos 2x}{2}dx=\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1+\cos 2x)dx=\frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)\Bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}=$$

$$=\frac{1}{2}\left(\frac{\pi}{4}+\frac{\sin\frac{\pi}{2}}{2}\right)-\frac{1}{2}\left(-\frac{\pi}{4}-\frac{\sin\frac{\pi}{2}}{2}\right)=\frac{1}{2}\left(\frac{\pi}{2}+1\right)=\frac{\pi+2}{4}.$$

**Example 4.7.** Calculate the integral:

$$\int_{3}^{-18}\sqrt[3]{2-\frac{x}{3}}dx.$$

*Solution.*

$$\int_{3}^{-18}\sqrt[3]{2-\frac{x}{3}}dx=\frac{1}{-\frac{1}{3}}\cdot\frac{\left(2-\frac{x}{3}\right)^{\frac{4}{3}}}{\frac{4}{3}}\Bigg|_{3}^{-18}=-\frac{9}{4}\left(\sqrt[3]{(2+6)^4}-\sqrt[3]{(2-1)^4}\right)=-\frac{135}{4}.$$

**Example 4.8.** Calculate the integral:

$$\int_{-\pi}^{\pi}(\sin 5x\cdot\cos 7x)dx.$$

*Solution.*

$$\int_{-\pi}^{\pi}(\sin 5x\cdot\cos 7x)dx=\frac{1}{2}\int_{-\pi}^{\pi}[\sin(5x+7x)+\sin(5x-7x)]dx=$$

$$=\frac{1}{2}\int_{-\pi}^{\pi}[\sin(12x)-\sin(2x)]dx=\frac{1}{2}\left(\frac{-\cos(12x)}{12}+\frac{\cos(2x)}{2}\right)\Bigg|_{-\pi}^{\pi}=$$

$$=\frac{1}{2}\left(\frac{-\cos(12\pi)}{12}+\frac{\cos(2\pi)}{2}\right)-\frac{1}{2}\left(\frac{-\cos(-12\pi)}{12}+\frac{\cos(-2\pi)}{2}\right)=$$

$$=\frac{1}{2}\left(\frac{-\cos(12\pi)}{12}+\frac{\cos(2\pi)}{2}\right)-\frac{1}{2}\left(\frac{-\cos(12\pi)}{12}+\frac{\cos(2\pi)}{2}\right)=0.$$

**Example 4.9.** Calculate the integral:

$$\int_0^{\frac{\pi}{4}}\left(e^{-4x}-\operatorname{tg}^2 x\right)dx.$$

*Solution.*

$$\int_0^{\frac{\pi}{4}}\left(e^{-4x}-\operatorname{tg}^2 x\right)dx=\int_0^{\frac{\pi}{4}}e^{-4x}\,dx-\int_0^{\frac{\pi}{4}}\frac{\sin^2 x}{\cos^2 x}dx=-\frac{1}{4}e^{-4x}\Big|_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\frac{1-\cos^2 x}{\cos^2 x}dx=$$

$$=-\frac{1}{4}\left(e^{-\pi}-e^0\right)-\int_0^{\frac{\pi}{4}}\left(\frac{1}{\cos^2 x}-1\right)dx=\frac{1}{4}\left(1-e^{-\pi}\right)-\left(\operatorname{tg}x-x\right)\Big|_0^{\frac{\pi}{4}}=$$

$$=\frac{1}{4}\left(1-e^{-\pi}\right)-\left(\operatorname{tg}\frac{\pi}{4}-\frac{\pi}{4}\right)+\left(\operatorname{tg}0-0\right)=\frac{1}{4}-\frac{1}{4e^{\pi}}-1+\frac{\pi}{4}=\frac{1}{4}\left(-3+\pi-\frac{1}{e^{\pi}}\right).$$

**Example 4.10.** Calculate the integral:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\cos 2x}{\cos^2 x\cdot\sin^2 x}dx.$$

*Solution.* $\int\limits_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x - \sin^2 x}{\cos^2 x \cdot \sin^2 x} dx = \int\limits_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} dx = \left(-\operatorname{ctg} x - \operatorname{tg} x\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} =$

$$= -\left(\operatorname{ctg}\frac{\pi}{3} + \operatorname{tg}\frac{\pi}{3}\right) + \left(\operatorname{ctg}\frac{\pi}{6} + \operatorname{tg}\frac{\pi}{6}\right) = -\left(\operatorname{tg}\frac{\pi}{6} + \operatorname{ctg}\frac{\pi}{6}\right) + \left(\operatorname{ctg}\frac{\pi}{6} + \operatorname{tg}\frac{\pi}{6}\right) = 0.$$

## 4.3. Calculation of the area of a figure

The concept of integral has a simple geometric interpretation, which makes it possible to calculate the area of different curved trapezoids (Fig. 4.1).

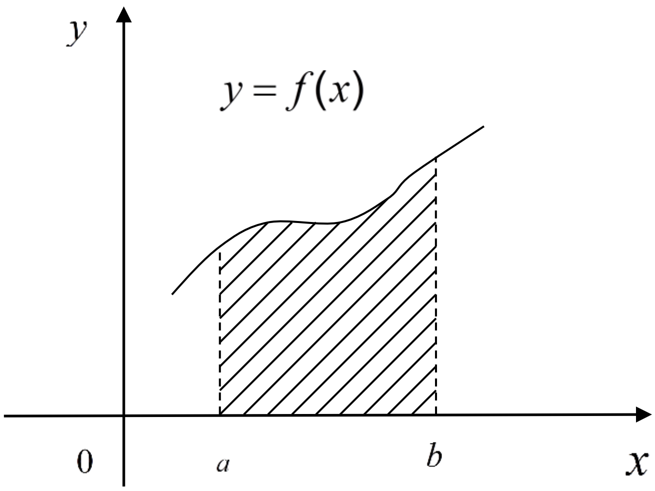



Fig. 4.1. **The area of a curved trapezoid**

The area of this figure is determined by the formula

$$S = \int\limits_a^b f(x)dx. \tag{4.7}$$

The area of the figure bounded by two continuous curves $y = f_1(x)$, $y = f_2(x)$ $\left(f_2(x) \geq f_1(x)\right)$ and the lines: $x = a$, $x = b$ (Fig. 4.2) can be calculated by the formula

$$S = \int\limits_a^b \left(f_2(x) - f_1(x)\right)dx, \tag{4.8}$$

regardless of the location of the curves relative to the abscissa.

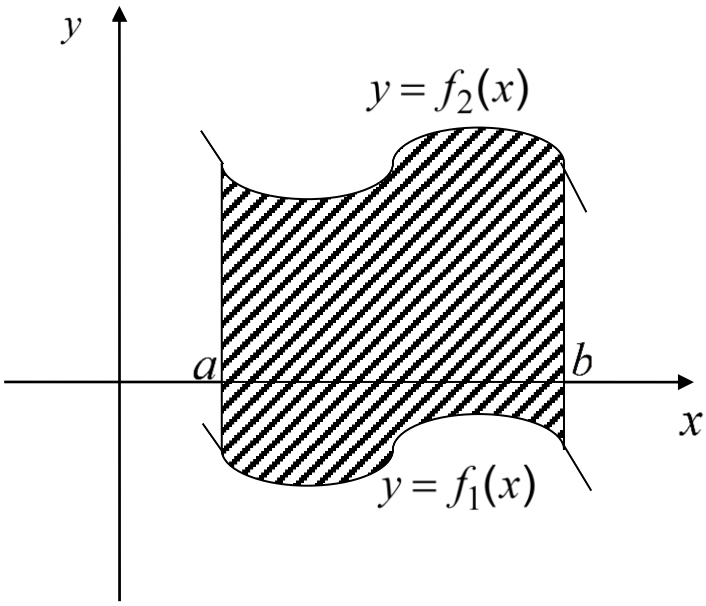



Fig. 4.2. **The area of the figure bounded by two continuous curves**

**Example 4.11.** Find the area of a figure bounded by the curve $y = e^{-4x}$, the lines $x = -0.25$, $x = 2$, and the abscissa.

*Solution.* Let's plot a graph for this task (Fig. 4.3). Using the formula (4.7), we find the needed area:

$$S = \int_{-0.25}^{4} e^{-4x} dx = \left.\frac{e^{-4x}}{-4}\right|_{-0.25}^{4} = \frac{e - e^{-8}}{4} (sq.\, units).$$

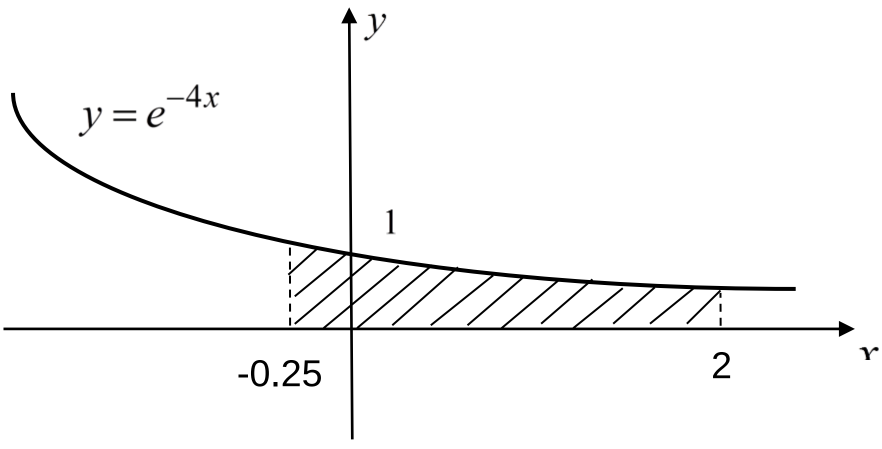



Fig. 4.3. **The area of the figure in example 4.11**

**Example 4.12.** Find the area of the figure bounded by the parabola $y = x^2 + 4x$ and the line $y = x + 4$.

*Solution.* Find the abscissae of the points of intersection of the parabola and the line, solving the system of equations:

$$\begin{cases} y = x^2 + 4x, \\ y = x + 4. \end{cases} \qquad (x_1 = -4, \quad x_2 = 1).$$

The area is shown in Fig. 4.4.

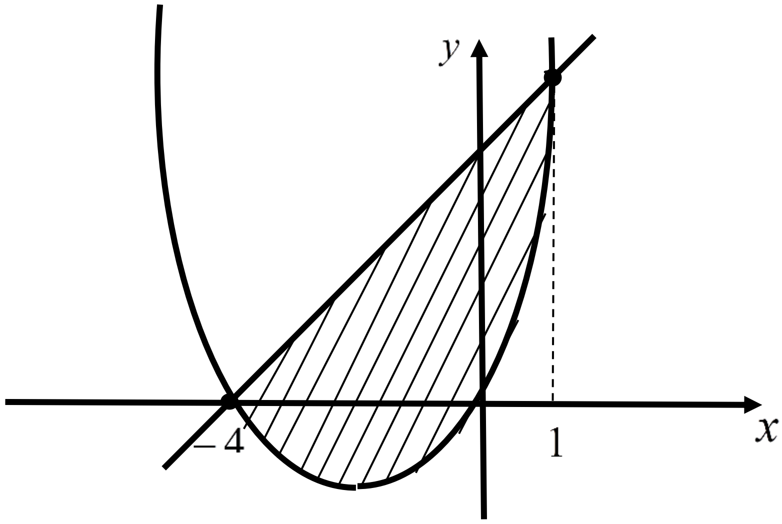



Fig. 4.4. **The area of the figure in example 4.12**

The sought area of the figure can be found by formula 4.8:

$$S = \int_{-4}^{1} \left( (x+4) - \left(x^2 + 4x\right) \right) dx = \int_{-4}^{1} \left( 4 - 3x - x^2 \right) dx = \left( 4x - \frac{3x^2}{2} - \frac{x^3}{3} \right)\Bigg|_{-4}^{1} =$$

$$= \left( 4 - \frac{3}{2} - \frac{1}{3} \right) - \left( -16 - 24 + \frac{64}{3} \right) = \frac{125}{6} \, (sq.\,units).$$

**Example 4.13.** Find the area of the figure bounded by the graph of the function $y = 2\sqrt{x}$ and the tangent drawn to the function $y = 1 + \ln x$ graph, which passes through the point with the abscissa $x_0 = 1$.

*Solution.* We find the equation of the tangent to the graph of the function $y = 1 + \ln x$ by the formula $y = y_0 + y'(x_0)(x - x_0)$. Insofar as

$x_0 = 1$ and $y_0 = 1 + \ln x = 1$; $y' = \frac{1}{x}$; $y'(x_0) = 1$, the equation of the tangent is $y = 1 + (x - 1)$ or $y = x$.

The sought area is shown in Fig. 4.5.

We find the abscissas of the intersection points of the constructed graphs: $x = 0$ and $x = 4$. The sought area is:

$$S = \int_0^4 \left(2\sqrt{x} - x\right)dx = \left.\left(2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{2}\right)\right|_0^4 = \frac{8}{3}(sq.\,units).$$

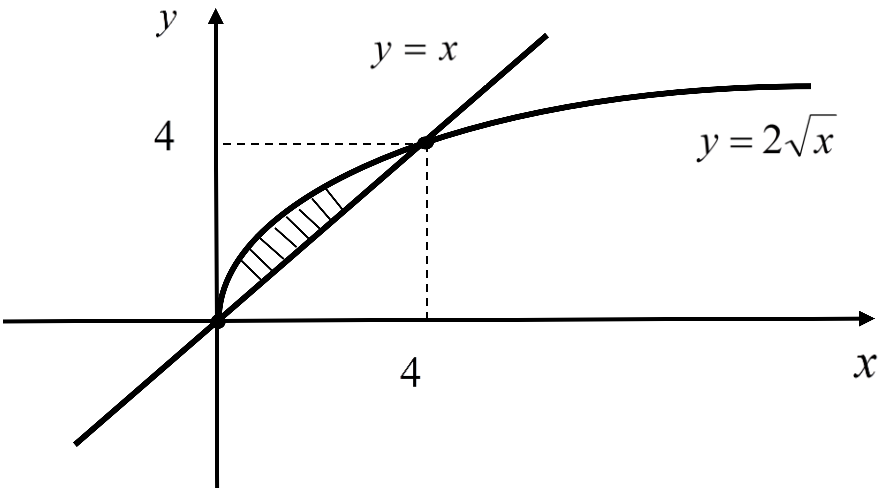



Fig. 4.5. **The area of the figure in example 4.13**

**Example 4.14.** Find the area of the figure bounded by the function graphs $y = \sqrt{1 + x}$, $y = \sqrt{7 - x}$ and $y = 0$.

*Solution.* After plotting the graphs of the functions $y = \sqrt{1 + x}$, and $y = \sqrt{7 - x}$ (Fig. 4.6), we find the abscissa of the intersection point from the equation:

$$\sqrt{1 + x} = \sqrt{7 - x}, x = 3.$$

The sought area is: $S = S_1 + S_2$. So,

$$S=\int_{-1}^{3}\sqrt{1+x}\,dx+\int_{3}^{7}\sqrt{7-x}\,dx=\int_{-1}^{3}(1+x)^{\frac{1}{2}}dx+\int_{3}^{7}(1+x)^{\frac{1}{2}}dx=$$

$$=\left.\frac{(1+x)^{\frac{3}{2}}}{\frac{3}{2}}\right|_{-1}^{3}-\left.\frac{(7-x)^{\frac{3}{2}}}{\frac{3}{2}}\right|_{3}^{7}=\frac{2}{3}(8+8)=\frac{32}{3}(sq.\,units).$$

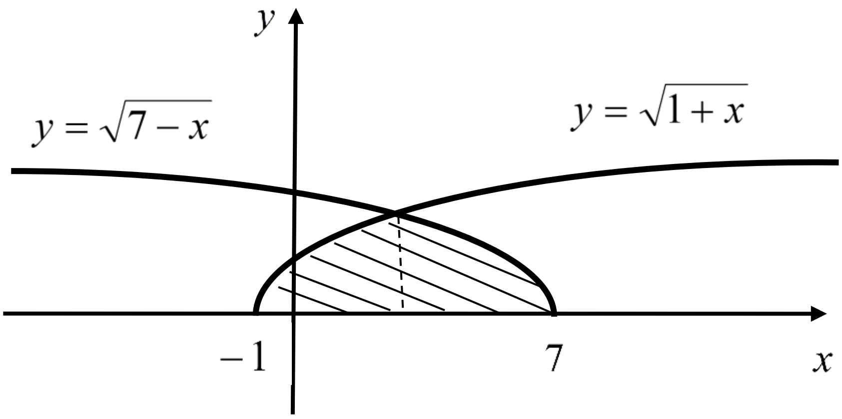



Fig. 4.6. **The area of the figure in example 4.14**

## Tasks for individual work

Prove that $F(x)$ is the integral of $f(x)$.

**Task 4.15.** $F(x)=\dfrac{(2x+5)^{10}}{20}+C$, $\quad f(x)=(2x+5)^{9}$.

**Task 4.16.** $F(x)=\dfrac{-\sqrt{(3-4x)^{3}}}{6}+C$, $\quad f(x)=\sqrt{3-4x}$.

**Task 4.17.** $F(x)=-\dfrac{\cos^{4}x}{4}+C$, $\quad f(x)=\sin x\cdot\cos^{3}x$.

**Task 4.18.** $F(x)=-\dfrac{2}{3}\cos^{3}x+C$, $\quad f(x)=\sin 2x\cdot\cos x$.

**Task 4.19.** $F(x)=\dfrac{1}{2}x-\dfrac{1}{8}\sin 4x+C$, $\quad f(x)=\sin^{2}2x$.

**Task 4.20.** $F(x)=\dfrac{1}{4}\sin 2x-\dfrac{1}{16}\sin 8x+C$, $\quad f(x)=\sin 3x\cdot\sin 5x$.

**Task 4.21.** $F(x)=\frac{1}{2}\ln(2x-1)+C, \quad f(x)=\frac{1}{2x-1}.$

**Task 4.22.** $F(x)=\frac{-\sqrt[3]{(1-3x)^4}}{20}+C, \quad f(x)=\sqrt[3]{1-3x}.$

**Task 4.23.** $F(x)=\ln x-\frac{1}{x}-\frac{1}{2x^2}+C, \quad f(x)=\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}.$

**Task 4.24.** $F(x)=2\sqrt{x}+\frac{2x\sqrt{x}}{3}+C, \quad f(x)=\frac{x+1}{\sqrt{x}}.$

For the given function $f(x)$, find the antiderivative function $F(x)$ whose graph goes through the point $M$.

**Task 4.25.** $f(x)=4x-4x^3$, $M(1;2)$.

**Task 4.26.** $f(x)=\frac{1}{2x-1}$, $M\left(-\frac{1}{2};\ln 2\right)$.

**Task 4.27.** $f(x)=\sqrt[3]{1-3x}$, $M(0;0)$.

**Task 4.28.** $f(x)=\frac{1}{2}\left(e^x+e^{-x}\right)$, $M(0;1)$.

**Task 4.29.** $f(x)=5x^4+3x^2-4$, $M(-1;12)$.

**Task 4.30.** $f(x)=\frac{3}{\sqrt{4x+5}}$, $M(5;4)$.

**Task 4.31.** $f(x)=\frac{3}{2\sqrt{x}}-2x$, $M(9;8)$.

**Task 4.32.** $f(x)=\frac{6}{\cos^2 6x}$, $M\left(\frac{\pi}{18};3\sqrt{3}\right)$.

**Task 4.33.** $f(x)=\frac{4}{\sin^2 4x}$, $M\left(\frac{\pi}{24};-2\sqrt{3}\right)$.

**Task 4.34.** $f(x)=\frac{5}{2\sqrt{x}}+x$, $M(4;-3)$.

**Task 4.35.** $f(x) = 16x^3 + e^{4x}$, $M\left(\frac{1}{2}; \frac{e^2}{4}\right)$.

**Task 4.36.** $f(x) = 4e^{2x-1} - 4$, $M(1;3e)$.

**Task 4.37.** $f(x) = \frac{1}{3}\sin\frac{x}{3} + 4\cos 4x$, $M(\pi; 3)$.

**Task 4.38.** $f(x) = \frac{1}{3}\sin\frac{x}{3} + 4\cos 4x$, $M(\pi; 3)$.

Find the indefinite integral of the given function $f(x)$.

**Task 4.39.** $f(x) = \sqrt[3]{x^2} + 2x\sqrt{x} - \frac{4}{x^2}$.

**Task 4.40.** $f(x) = 3e^{-\frac{x}{3}} + \cos 5x$.

**Task 4.41.** $f(x) = \frac{x^4 - 3x + 2}{x^5}$.

**Task 4.42.** $f(x) = \frac{1}{\cos^2(2x-1)}$.

**Task 4.43.** $f(x) = 2e^{-2x+5}$.

**Task 4.44.** $f(x) = \frac{\sqrt{x} - x^3 e^x + x^2}{x^3}$.

**Task 4.45.** $f(x) = \frac{\cos 2x}{\cos^2 x \cdot \sin^2 x}$.

**Task 4.46.** $f(x) = \mathrm{tg}^2 x$.

**Task 4.47.** $f(x) = \frac{1}{\cos^2 x \cdot \sin^2 x}$.

**Task 4.48.** $f(x) = \frac{1 + \cos 2x}{\cos x}$.

**Task 4.49.** $f(x) = \cos^2 2x$.

**Task 4.50.** $f(x) = \frac{2}{(x+3)^2} + \frac{4}{\sin^2 2x}$.

Calculate the definite integrals.

**Task 4.51.** $\int_{-1}^{3} x^3 dx$; **Task 4.52.** $\int_{1}^{2}\left(x^2+\frac{1}{x^2}\right)dx$;

**Task 4.53.** $\int_{1}^{16}\sqrt{x}\cdot dx$; **Task 4.54.** $\int_{\frac{1}{e}}^{\frac{1}{\sqrt{e}}}\frac{dx}{x}$;

**Task 4.55.** $\int_{1}^{8}\left(4x-\frac{1}{3\sqrt[3]{x^2}}\right)dx$; **Task 4.56.** $\int_{0}^{1}\sqrt{1-x}dx$;

**Task 4.57.** $\int_{1}^{4}\frac{1+\sqrt{x}}{x^2}dx$; **Task 4.58.** $\int_{0}^{8}(\sqrt{2x}+\sqrt[3]{x})dx$;

**Task 4.59.** $\int_{0}^{\frac{\pi}{2}}\sin 3x\cdot dx$; **Task 4.60.** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos 3x\cdot\cos 5x dx$;

**Task 4.61.** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin 2x\cdot\sin 7x dx$; **Task 4.62.** $\int_{0}^{2}(1-3x)^{-4}dx$;

**Task 4.63.** $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\frac{2}{\sin^2 x}+\frac{3}{\cos^2 x}\right)dx$; **Task 4.64.** $\int_{0}^{\pi}\cos^2 x\cdot dx$;

**Task 4.65.** $\int_{0}^{\frac{\pi}{6}}\frac{\sin 2x}{\sin x}dx$; **Task 4.66.** $\int_{1}^{0.5}\left(4x-\frac{1}{2x}\right)dx$;

**Task 4.67.** $\int_{0}^{\ln 2}\left(e^x+e^{-x}\right)dx$; **Task 4.68.** $\int_{0}^{13}\frac{dx}{\sqrt[3]{(1+2x)^2}}$.

Find the area of the figure bounded by the given lines.

**Task 4.69.** $y = x^2 + 2x$, $y = 0$.
**Task 4.70.** $xy = 4$, $x = 1, x = 4, y = 0$.
**Task 4.71.** $y = x^2$, $y^2 = 8x$.
**Task 4.72.** $y = \sqrt{x}$, $y = 0, x = 9$.
**Task 4.73.** $y = -x^2 - x + 2$, $y = 0$.
**Task 4.74.** $y = -2(x-3)^2 + 2$, $y = 0$.
**Task 4.75.** $xy = 4$, $x + y = 5$.
**Task 4.76.** $y = x^2$, $y = 2x - x^2$.
**Task 4.77.** $y = x^2 - 3x$, $y = 4 - 3x$.
**Task 4.78.** $y = (x-1)^2 + 2x$, $y = x + 1$.
**Task 4.79.** $y = \sqrt{x}$, $y = \sqrt{4-3x}$, $y = 0$.
**Task 4.80.** $y = \sqrt{x}$, $y = 2 - x^2$, $x = -1$, $y = 0$.
**Task 4.81.** $y = 1 - 2x$, $y = \dfrac{1}{x+1}$.

**Task 4.82.** Calculate the area of the shape bounded by the curve $y = x^2 - x + 2$ and the tangent drawn to the curve $y = 3 + \ln x$ at the point with the abscissa $x_0 = 1$.

**Task 4.83.** Calculate the area of the figure bounded by the parabola $y = -x^2 + 4x - 3$ and the tangents to it at the points $M(0;3)$ and $N(3;0)$.

**Task 4.84.** Calculate the area of the figure bounded by the parabola $y = -x^2 + 4x - 3$, and the tangent to it at the point $M(3;5)$, and the ordinate axis.

**Task 4.85.** Calculate the area of the figure bounded by the parabola $y = -x^2 + 4x - 3$, tangents to it at the point $x = 2$ and the line $x = 1$.

## Questions for self-assessment

1. What is the antiderivative of the function $f(x)$?

2. What is the integration called?

3. Name the elements of the table of antiderivatives.

4. What is the definite integral of $f(x)$ with lower ($a$) and upper ($b$) limits called?

5. Name the first fundamental theorem of calculus.

6. Name the simplest properties of the definite integrals.

7. What is a simple geometric interpretation of an integral?

8. Name the formula that determines the area of a curved trapezoid.

9. What is the area of the figure bounded by the two continuous curves $y = f_1(x)$, $y = f_2(x)$ and the lines: $x = a$, $x = b$.

10. How is the antiderivative function $F(x)$ for the given function $f(x)$, whose graph goes through the point $M$ defined?

# Answers

## Section 1

**1.3.** 1) $\sqrt{2}-\sqrt{3}$; 2) $-0.5$; 3) $-\frac{7}{3}$. **1.4.** 1) $2ab$; 2) $(a-b)^2$; 3) $a+b$. **1.8.** 1) $(+)$; 2) $(+)$; 3) $(-)$; 4) $(-)$; 5) $(+)$; 6) $(+)$; 7) $(+)$; 8) $(-)$. **1.9.** 1) $(+)$; 2) $(+)$; 3) $(-)$; 4) $(-)$; 5) $(+)$; 6) $(+)$. **1.10.** 1) $0$. 2) $-\frac{5}{2}$. 3) $-2$. 4) $2$. **1.11.** 1) $a^2+b^2$; 2) $a^3+b^3$; 3) $\frac{(a+b)(a^2+b^2)}{a-b}$. **1.12.** 1) $(+)$; 2) $(-)$; 3) $(-)$; 4) $(-)$; 5) $(+)$; 6) $(-)$; 7) $(+)$; 8) $(+)$; 9) $(+)$. **1.18.** 1) $\cos\alpha = \frac{12}{13}$; $\operatorname{tg}\alpha = -\frac{5}{12}$; 2) $\sin\alpha = \frac{15}{17}$; $\operatorname{tg}\alpha = -\frac{15}{8}$; 3) $\operatorname{ctg}\alpha = \frac{13}{5}$; $\sin\alpha = -\frac{5}{\sqrt{194}}$; $\cos\alpha = -\frac{13}{\sqrt{194}}$; 4) $\operatorname{tg}\alpha = -\frac{24}{7}$; $\sin\alpha = \frac{24}{25}$; $\cos\alpha = \frac{7}{25}$. **1.19.** $-\frac{5}{12}$. **1.20.** $0.6$. **1.21.** $\frac{\sqrt{3}}{3}$. **1.22.** $\frac{8}{7}$. **1.23.** 1) $\frac{1}{\cos^2\alpha}$; 2) $\sin^2\alpha$; 3) $2$;

4) $\dfrac{2}{\sin\alpha}$; 5) $\dfrac{1}{\cos\alpha}$; 6) $\dfrac{2}{\cos\alpha}$. **1.30.** 1) $0$; 2) $2\sin 22^\circ$; 3) $\sin\dfrac{2}{15}$; 4) $\dfrac{1}{2}$.

**1.31.** 1) $-\dfrac{117}{125}$; 2) $\dfrac{416}{425}$; 3) $\dfrac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}$; 4) $2-\sqrt{3}$. **1.32.** 1) $-\sin 2\alpha$;

2) $\sqrt{3}\cos\alpha$; 3) $-2\sin\alpha\sin\beta$; 4) $\operatorname{tg}\alpha\cdot\operatorname{tg}\beta$; 5) $\operatorname{tg}(\alpha+\beta)$. **1.33.** 1) $\dfrac{\sqrt{6}}{3}$;

2) $\sqrt{2}\sin\left(3\alpha+\dfrac{\pi}{4}\right)$; 3) $\sqrt{2}$; 4) $\operatorname{tg}10^\circ$; 5) $2\cos\left(\alpha-\dfrac{\pi}{6}\right)$. **1.40.** 1) $\dfrac{\sqrt{2}}{2}$;

2) $-\sqrt{3}$; 3) $-\dfrac{1}{2}$; 4) $-\operatorname{tg}40^\circ$; 5) $2$; 6) $0$; 7) $0$. **1.41**. 1) $2\operatorname{tg}\alpha$; 2) $\dfrac{1}{\sin^2\alpha}$;

3) $\sin 6\alpha$; 4) $1$; 5) $-\cos^2\alpha$; 6) $-\dfrac{\sin^2\alpha}{\operatorname{tg}2\alpha}$; **1.53.** 1) $1$; 2) $4\sin\alpha$; 3) $0.5\sin 4\alpha$;

4) $\dfrac{1}{\cos\alpha}$; 5) $\dfrac{1}{2}\sin 8\alpha$; 6) $0$; 7) $2\sin\alpha$; 8) $4\cos\alpha$; 9) $-1$; 10) $2$;

11) $\sin^2 2\alpha$. 12) $\operatorname{tg}4\alpha$ 13) $\operatorname{tg}\alpha$; **1.54.** 1) $\dfrac{24}{25}; -\dfrac{7}{25}; -\dfrac{24}{7}$; 2) $\dfrac{1}{2}; \dfrac{\sqrt{3}}{2}; \dfrac{\sqrt{3}}{3}$;

3) $\dfrac{1}{7}$; 4) $1{,}5$; 5) $0{,}5$; 6) $\dfrac{1}{\sqrt{3}}$; 7) $-\dfrac{1}{2}\sin\dfrac{\pi}{5}$ 8) $12$; 9) $\operatorname{tg}\dfrac{\alpha}{2}$. **1.63.** 1) $0$;

2) $-\sin 18^\circ$; 3) $\sqrt{2}\sin 25^\circ$; 4) $-\sqrt{3}$; 5) $4\sin 3\alpha\cdot\cos\dfrac{\alpha}{2}\cdot\cos\dfrac{3\alpha}{2}$;

6) $-\operatorname{tg}5\alpha\cdot\operatorname{tg}\alpha$; 7) $\operatorname{tg}\alpha\cdot\operatorname{ctg}5\alpha$; 8) $-\operatorname{tg}^2\dfrac{\alpha}{2}$; 9) $4\cos 1^\circ\cos 2^\circ\cos 25^\circ$; 10) $4$;

11) $4\cos\left(30^\circ+\dfrac{\alpha}{2}\right)\cos\left(30^\circ-\dfrac{\alpha}{2}\right)$; 12) $4\sin\left(\dfrac{\pi}{8}+\dfrac{\alpha}{2}\right)\cos\left(\dfrac{\pi}{8}-\dfrac{\alpha}{2}\right)$;

13) $2\sin\left(60^\circ+\alpha\right)$; **1.64.** 1) $\sin\alpha+\sin 2\alpha+\sin 3\alpha+\sin 4\alpha$; 2) $\cos\alpha+$

$+\cos 2\alpha+\cos 3\alpha+\cos 4\alpha$; 3) $\dfrac{1}{4}\sin 2\alpha-\dfrac{1}{8}\sin 4\alpha$; 4) $\dfrac{1}{2}\left(\cos 4\alpha-\cos 6\alpha\right)$;

5) $-\dfrac{1}{2}\cos 2\alpha$; 6) $1+2\sin 2\alpha$. **1.65.** 1) $\dfrac{\sin 3\alpha}{\sin 5\alpha}$; 2) $\operatorname{tg}2\alpha$; 3) $\operatorname{tg}3\alpha$; 4) $2\sin 2\alpha$;

5) $\cos 2\alpha$; 6) $\cos^2\dfrac{\alpha}{2}$; 7) $\sin 3\alpha$; 8) $4\cos\alpha\cos 2\alpha\cos 3\alpha$; **1.70.** 1) $\dfrac{4\pi}{3}$;

2) $\frac{\pi}{3}$; 3) $1.5\pi$; 4) $\frac{5\pi}{12}$; 5) $\frac{13\pi}{12}$. **1.71.** 1) 0; 2) $\sqrt{3}$; 3) 0; 4) 1; 5) $-1$.

**1.72.** 1) $\frac{\pi}{10}+\frac{2\pi n}{5}, n\in Z$; 2) $(-1)^n\frac{\pi}{30}+\frac{\pi n}{10}, n\in Z$; 3) $\pm\frac{5\pi}{6}+2\pi n$; 4) $\frac{\pi}{3}+\pi n$, $n\in Z$; 5) $\frac{\pi}{12}+\frac{\pi n}{2}$, $n\in Z$; 6) $-\frac{\pi}{8}+\pi n, n\in Z$; 7) $\frac{\pi n}{3}, n\in Z$. **1.80.** 1) $x_1=\frac{\pi}{2}+2\pi n$, $x_2=(-1)^n\frac{\pi}{6}+\ +\pi n, n\in Z$; 2) $x=\frac{\pi}{2}+2\pi n, n\in Z$; 3) $x_1=\frac{1}{2}\left(\operatorname{arctg}7+\frac{\pi n}{2}\right)$, $x_2=\frac{\pi n}{2}, n\in Z$; 4) $x_1=-\frac{\pi}{2}+2\pi n$, $x_2=(-1)^n\frac{\pi}{6}+\pi n, n\in Z$; 5) $x_1=\pi+2\pi n$, $x_2=\pm\arccos\frac{3}{4}+2\pi n, n\in Z$. **1.81.** 1) $-\frac{\pi}{4}+\pi n, n\in Z$; 2) $x_2=\operatorname{arctg}2+\pi n$, $x_1=-\operatorname{arctg}\frac{1}{3}+\pi n, n\in Z$. 3) $x_1=\operatorname{arctg}2+\pi n$, $x_2=-\operatorname{arctg}\frac{3}{4}+\pi n, n\in Z$.

**1.82.** 1) $x_1=\pi n, x_2=\frac{\pi}{4}+\frac{\pi n}{2}, n\in Z$; 2) $x_1=2\pi n$, $x_2=\frac{\pi}{5}+\frac{2\pi n}{5}, n\in Z$; 3) $x_1=\frac{2}{3}\pi n$, $x_2=\pm\frac{\pi}{4}+\pi n$, $n\in Z$; 4) $x_1=\frac{\pi}{3}+\frac{\pi n}{2}, x_2=\pi n$, $x_3=\frac{7\pi}{24}+\frac{\pi n}{2}$, $n\in Z$; 5) $x_1=2\pi n, x_2=\frac{\pi}{2}+2\pi n$, $x_3=\frac{\pi}{5}+\frac{4\pi n}{5}$, $n\in Z$; 6) $x_1=\frac{\pi n}{2}$, $x_2=(-1)^n\frac{\pi}{21}+\frac{\pi n}{7}$, $n\in Z$; 7) $x_1=\frac{\pi}{24}+\frac{\pi n}{2}$, $x_2=-\frac{\pi}{12}+\frac{\pi n}{2}, n\in Z$.

**1.83.** 1) $x_1=\frac{\pi}{16}+\frac{\pi n}{8}$, $x_3=\frac{\pi}{10}+\frac{\pi n}{5}$, $n\in Z$; 2) $x_3=\frac{\pi}{4}+\pi n$, $n\in Z$; 3) $x_1=\pi n$; $x_2=\frac{\pi}{4}+\frac{\pi n}{2}$, $n\in Z$; **1.84.** 1) $x_1=\frac{\pi n}{4}, n\in Z$; 2) $x_1=\frac{\pi n}{3}$, $x_2=\pi n$; $x_3=\frac{\pi n}{2}, n\in Z$; 3) $x_1=\frac{\pi n}{5}$; $x_2=\frac{\pi}{2}+\frac{\pi n}{2}, n\in Z$; 4) $x=\frac{\pi}{2}+\pi n$, $n\in Z$. **1.85.** 1) $x_1=\frac{\pi n}{4}$, $n\in Z$; 2) $x_1=-\operatorname{arctg}\frac{1}{3}+\pi n$, $n\in Z$; $x_2=2\operatorname{arctg}\frac{1}{2}+2\pi n, n\in Z$; 3) $x_1=\pi+2\pi n$, $x_2=2\operatorname{arctg}\frac{2}{3}+2\pi n, n\in Z$;

4) $\varnothing$. **1.86.** 1) $x_1 = 2\pi n,\quad x_2 = \frac{\pi}{2} + 2\pi n, n \in Z;$ 2) $x_1 = \frac{\pi}{2} + 2\pi n, x_2 = 2\pi n,$ $x_2 = 2\,\mathrm{arctg}\left(-5 \pm \sqrt{14}\right) + 2\pi n, n \in Z;$ 3) $x_1 = \frac{\pi}{2} + 2\pi n,\quad x_2 = \pi + 2\pi n, n \in Z$.

**1.87.** 1) $x = \frac{\pi}{2} + 2\pi n,\ n \in Z;$ 2) $x_1 = \pi + 2\pi n,\ x_2 = \pm\arccos\frac{3}{4} + 2\pi n, n \in Z;$ 3) $x = (-1)^{n+1}\frac{\pi}{12} + \frac{\pi n}{2},\quad n \in Z;$ 4) $x = \pi n, n \in Z;$ 5) $x_1 = \frac{\pi}{8} + \frac{\pi n}{3},$ $x_2 = \frac{\pi}{16} + \frac{\pi n}{2}, n \in Z;$ 6) $x_1 = \frac{\pi}{4} + \frac{\pi n}{2}, n \in Z;$ 7) $x = (-1)^n\frac{\pi}{18} + \frac{\pi n}{3},\quad n \in Z;$ 8) $x_1 = \frac{\pi}{4} + 2\pi n,\quad x_2 = (-1)^n\frac{\pi}{24} + \frac{\pi n}{4},\quad n \in Z;$ 9) $x = \frac{\pi}{4} + \pi n,\quad n \in Z;$ 10) $x_1 = \frac{\pi}{9} + \frac{2}{9}\pi n,\quad x_2 = \pi + 2\pi n, n \in Z;$ 11) $x = \frac{\pi}{8} + \frac{\pi n}{4}, x_2 = \frac{\pi n}{2}\quad n \in Z;$ 12) $x_1 = \frac{\pi}{4} + \pi n, x_2 = \mathrm{arctg}\,2 + \pi n,\quad n \in Z;$ 13) $x_1 = \frac{\pi}{2} + 2\pi n,\quad x_2 = 2\pi n,$ $x_3 = \frac{\pi}{4} \pm \arccos\frac{7}{5\sqrt{2}} + 2\pi n,\ n \in Z;$ 14) $x_1 = -\frac{\pi}{2} + 2\pi n,\ x_2 = \pm\frac{\pi}{3} + \pi n, n \in Z$.

**1.89.** 1) $\begin{cases} x_1 = \frac{5\pi}{6} + 2\pi n, \\ y_1 = \frac{\pi}{6} - 2\pi n, \end{cases}\quad \begin{cases} x_2 = \frac{\pi}{6} + 2\pi n, \\ y_2 = \frac{5\pi}{6} - 2\pi n, \end{cases}\quad n \in Z.$

2) $\begin{cases} x_1 = \pi n + \frac{\pi}{4}, \\ y_1 = -\pi n, \end{cases}\quad \begin{cases} x_2 = \pi n, \\ y_2 = -\pi n + \frac{\pi}{4}, \end{cases}\quad n \in Z.$

3) $\begin{cases} x_1 = -\frac{\pi}{6} + \pi(n+k), \\ y_1 = -\frac{\pi}{6} + \pi(n-k), \end{cases}\quad \begin{cases} x_2 = \frac{\pi}{6} + \pi(n+k), \\ y_2 = \frac{\pi}{6} + \pi(n-k), \end{cases}\quad n \in Z,\ k \in Z.$

4) $\begin{cases} x_1 = \frac{\pi}{6} + \pi(n+k), \\ y_1 = -\frac{\pi}{6} - \pi(n-k), \end{cases}\quad \begin{cases} x_2 = -\frac{\pi}{6} + \pi(n+k), \\ y_2 = \frac{\pi}{6} - \pi(n-k). \end{cases}\quad n \in Z,\ k \in Z.$

## Section 2

**2.4.** 1) $48$; 2) $36$; 3) $20$; 4) $3$; 5) $24$; 6) $12{,}5$; 7) $4$; 8) $\frac{2(m+2)}{2-m}$; 9) $\frac{3(1-a)}{1+b}$.

**2.6.** 1) $\left(-\infty;\frac{3}{2}\right)$; 2) $(-8;4)$; 3) $(-\infty;-2)\cup(2;\infty)$; 4) $(-\infty;0)\cup(2;\infty)$.

**2.13.** 1) $2;6$; 2) $-1;3$; 3) $4$; 4) $8$; 5) $15;120$; 6) $8$; 7) $4$; 8) $4;7$; 9) $81$; 10) $5(5\pm\sqrt{53})$; 11) $1\pm\sqrt{10}$. **2.14.** 1) $2$; 2) $9$; 3) $3$; 4) $16$; 5) $4$; 6) $1$; 7) $-1$. **2.15.** 1) 2; 2) 0; 3) $36$; 4) $-0.25; 0.75$; 5) $\pm 1$; 6) $2$; 7) $9$; 8) $3;11$; 9) $-1;4$. **2.16** 1) $\{0;1\}$; 2) $\{-2;-1\}$; 3) $\{-2\}$; 4) $\left\{\log_{\frac{5}{2}}2\right\}$; 5) $\{-1;0\}$; 6) $\{-2;2\}$. **2.22.** 1) $2$; 2) $4$; 3) $3+\sqrt[3]{7}$; 4) $3$; 5) $5$; 6) $\{25\}$; 7) $\{2\}$; 8) $\{11;9\}$; **2.23.** 1) $\{86;7\}$; 2) $\{14\}$; 3) $\{2;9\}$; 4) $\{13\}$; 5) $\{5\}$.

**2.24.** 1) $10$; 2) $100$; 3) $\{1+\sqrt[3]{10};1001\}$; 4) $\{-0{,}5\}$; 5) $\{\sqrt[4]{10};10\}$; 6) $\{-2;-8\}$.

**2.25.** 1) $\frac{1}{16}; 4$; 2) $0{,}1; 1000$; 3) $0{,}01; 10$; 4) $0{,}01; 100$; 5) $3; 27$; 6) $\frac{1}{2}; 4$.

**2.26.** 1) $5$; 2) 126; 3) $\{27\}$; 4) $\{16\}$; 5) $\{\sqrt{2};4\}$; 6) $\left\{-\frac{5}{4}\right\}$; 7) $\{1;2\}$; 8) $\left\{\frac{1}{6};6\right\}$; 9) $\left\{\frac{1}{9};9\right\}$. **2.32.** 1) $\left(2;\frac{3}{2}\right)$; 2) $(1;10)$; 3) $(1;1)$; 4) $(5;2)$; 5) $(2;2)$; 6) $(3;2)$; 7) $(2;4)$; $(4;2)$; 8) $(125;4)$; $(625;3)$; 9) $(3;2)$; $(2;3)$; 10) $(18;2)$; $(2;18)$.

**2.39.** 1) $(-\infty;1)$; 2) $\left]-\infty;\frac{7}{8}\right]$; 3) $(-3;1)$; 4) $\left(-\infty;\frac{1}{4}(1-\sqrt{17})\right)\cup\left(\frac{1}{4}(1+\sqrt{17});\infty\right)$; 5) $(1;4)$; 6) $(-\infty;-2)\cup\left(-\frac{2}{5};\infty\right)$; 7) $(-1;3)$; 8) $[1;5]$; 9) $(-\infty;0)$; 10) $(2;\infty)$; 11) $(-\infty;1)$; 12) $[0;1)\cup(3;\infty)$.

**2.40.** 1) $(2;\infty)$; 2) $(-\infty;4)$; 3) $[3;\infty)$; 4) $(-\infty;7)$; 5) $(-\infty;2)$; 6) $[0;16]$; 7) $(-\infty;\log_2 3]$; **2.47.** 1) $\left(\frac{2}{5};\frac{3}{7}\right]$; 2) $(-2;-1]$; 3) $(-3;2)$; 4) $\left(\frac{1}{3};1\right)$; 5) $(50;\infty)$; 6) $(1;2)\cup(3;4)$; 7) $(2;3)$; 8) $(4;6)$; 9) $[-1;0)$; 10) $(1;\infty)$.

**2.48.** 1) $[2;3)$; 2) $[-7;3]$; 3) $(2;3)\cup[5;\infty)$; 4) $(3;4)\cup(6;\infty)$; 5) $(3;4)\cup[5;\infty)$;

6) $(-\infty;0)\cup(1;2)\cup(2;3)\cup(4;\infty)$. **2.49.** 1) $\left(\frac{2}{3};1\right)\cup(1;\infty)$; 2) $(0.5;1)\cup[2;\infty)$; 3) $(0;1)\cup(1;2)\cup(2;\infty)$. **2.50.** 1) $(-\infty;0)$; 2) $[0;\infty)$; 3) $(-\sqrt{2};-1)\cup(1;\sqrt{2})$; 4) $(0;0{,}01)\cup(100;\infty)$. **2.51.** 1) $(-5;0)\cup(3;\infty)$; 2) $(0;1)$; 3) $\left(0;\frac{3}{2}\right)$; 4) $(1;4]$; 5) $[2;3]$; 6) $(2;\infty)$; 7) $\left(-3;-\frac{2}{3}\right]$.

## Section 3

**3.12.** $\infty$. **3.13.** $6$. **3.14.** $\frac{1}{12}$. **3.15.** $2$. **3.16.** $\frac{3}{2}$. **3.17.** $1$. **3.18.** $\infty$. **3.19.** $4$. **3.20.** $-\frac{1}{3}$. **3.21.** $4$. **3.22.** $\frac{1}{2}$. **3.23.** $\infty$. **3.24.** $0$. **3.25.** $0$. **3.26.** $-3$. **3.33.** $10x-2+5e^{x}$. **3.34.** $7x^{\frac{5}{2}}+\frac{3}{2}x^{\frac{1}{2}}$. **3.35.** $42x+1$. **3.36.** $20x^4+4x^3-8x^2-4x$. **3.37.** $\frac{-20x^2+30x-12}{(3-4x)^2}$. **3.38.** $30x\left(3x^2+4\right)^4$. **3.39.** $4\left(5x^3-\frac{1}{x}\right)^3\left(15x^2+\frac{1}{x^2}\right)$. **3.40.** $\frac{2x}{\sqrt{1+2x^2}}$. **3.41.** $6x/3\sqrt[3]{\left(8x^2-1\right)^2}$. **3.42.** $\frac{1-\cos x-x\sin x}{(1-\cos x)^2}$. **3.43.** $6x\cos x^2$. **3.44.** $6\sin^2 x\cos x$. **3.45.** $-9\sin(3x-5)$. **3.46.** $-\frac{1}{x^2\cos^2\frac{1}{x}}$. **3.47.** $\frac{-2\operatorname{ctg}x}{\sin^2 x}$. **3.48.** $\frac{2\arcsin x}{\sqrt{1-x^2}}$. **3.49.** $\frac{2}{2x+1}$. **3.50.** $\operatorname{ctg}x$. **3.51.** $\frac{3\ln^2(x+2)}{x+2}$. **3.52.** $2\cdot 10^{2x-1}\ln 10$. **3.53.** $\frac{e^{\sqrt{x+1}}}{2\sqrt{x+1}}$. **3.54.** $3^{\sin 2x}2\ln 3\cos 2x$. **3.55.** $e^{\sqrt{\ln x}}\cdot\frac{1}{2x\sqrt{\ln x}}$. **3.56.** $\frac{2e^{\arcsin 2x}}{\sqrt{1-4x^2}}$. **3.60.** $6x-y-4=0$. **3.61.** $y=-5x-3$. **3.62.** $y=-3x+\frac{\pi}{2}$. **3.67.** $y$ increases on $x\in(-\infty;0)\cup$

$\cup(3;\infty)$ and decreases on $x\in(-1;3)$. **3.68.** $y$ increases on $x\in(-\infty;0)\cup$ $\cup(1;\infty)$ and decreases on $x\in(0;1)$. **3.69.** $y$ increases on $x\in\left(\frac{1}{2};\infty\right)$ and decreases on $x\in\left(0;\frac{1}{2}\right)$. **3.70.** $y$ increases on $x\in(-\infty;-1)\cup(3;\infty)$. **3.71.** $y_{\max}=17$ at $x=4$; $y_{\min}=-3$ at $x=1$ and $x=2$. **3.72.** $y_{\max}=20$ at $x=\pm3$; $y_{\min}=-12$ at $x=1$. **3.73.** $y_{\max}=e$ at $x=e$; $y_{\min}=0$ at $x=1$. **3.74.** there is no $y_{\max}$; $y_{\min}=-\frac{\pi}{2}$ at $x=1$. **3.75.** $x==\frac{13}{2}$; $y=-\frac{13}{2}$. **3.76.** $x=\frac{\sqrt{5}R}{5}$; $y=\frac{4\sqrt{5}R}{5}$.

## Section 4

**4.25.** $2x^2-x^3+1.$ **4.26.** $y=\frac{1}{2}\ln|4x-2|.$ **4.27.** $\frac{1-\sqrt[3]{(1-3x)^4}}{4}.$

**4.28.** $\cdot\frac{1}{2}(e^x-e^{-x})+1.$ **4.29.** $x^5+x^3-4x+10.$ **4.30.** $\frac{3}{2}\sqrt{4x+5}-\frac{7}{2}.$

**4.31.** $3\sqrt{x}-x^2+80.$ **4.32.** $\mathrm{tg}\,6x+2\sqrt{3}.$ **4.33.** $-\mathrm{ctg}\,4x-\frac{5}{\sqrt{3}}.$

**4.34.** $5\sqrt{x}+\frac{x^2}{2}-21.$ **4.35.** $4x^4+\frac{1}{4}e^{4x}-\frac{1}{4}.$ **4.36.** $2e^{2x-1}+e.$

**4.37.** $-\cos\frac{x}{3}+\sin 4x+3.5.$ **4.38.** $\sin\frac{x}{2}+\cos 5x.$ **4.39.** $\frac{3}{5}\sqrt[3]{x^5}+\frac{4}{5}\sqrt{x^5}+\frac{4}{x}+c.$

**4.40.** $-9e^{-\frac{x}{3}}+...+\frac{1}{5}\sin 5x+c.$ **4.41.** $\ln|x|+\frac{1}{x^3}-\frac{1}{2x^4}+c.$

**4.42.** $\frac{1}{2}\mathrm{tg}(2x-1)+c.$ **4.43.** $\frac{-1}{e^{2x-5}}+c.$ **4.44.** $-\frac{2}{3}\cdot\frac{1}{\sqrt{x^3}}-e^x+\ln|x|+c.$

**4.45.** $-\mathrm{tg}x-\mathrm{ctg}x+c.$ **4.46.** $\mathrm{tg}x-x+c.$ **4.47.** $\mathrm{tg}x-\mathrm{ctg}x+c.$

**4.48.** $2\sin x+c.$ **4.49.** $\frac{1}{2}x+\frac{1}{8}\sin 4x+c.$ **4.50.** $-\frac{2}{x+3}-2\mathrm{ctg}\,2x+c.$

**4.51.** 20. **4.52.** $\frac{17}{6}$. **4.53.** 42. **4.54.** $-\frac{3}{2}$. **4.55.** 125. **4.56.** $\frac{2}{3}$. **4.57.** $\frac{7}{4}$.

**4.58.** $\frac{100}{3}$. **4.59.** $\frac{1}{3}$ **4.60.** 0. **4.61.** $\frac{4}{45}$. **4.62.** $-\frac{14}{125}$. **4.63**. $\sqrt{3}+1$. **4.64.** $\frac{\pi}{2}$.

**4.65.** 1. **4.66.** $\frac{\ln 2-3}{2}$. **4.67.** $1.5$. **4.68.** 3. **4.69.** $4/3$. **4.70.** $8\ln 2$. **4.71.** $8/3$.

**4.72.** 18. **4.73.** 4.5. **4.74.** $8/3$. **4.75.** $7.5-8\ln 2$. **4.76.** $1/3$. **4.77.** $32/3$. **4.78.** 4.5.

**4.79.** $8/9$. **4.80.** $\frac{3}{4}-\ln 2$. **4.81.** $\frac{4\sqrt{2}}{3}-1$. **4.82.** $4/3$. **4.83.** $9/4$. **4.84.** 9.

**4.85.** $\ln 2-\frac{5}{8}$.

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# ЕЛЕМЕНТАРНА МАТЕМАТИКА (ТРИГОНОМЕТРІЯ ТА ПОЧАТОК АНАЛІЗУ)

**Навчальний посібник для слухачів підготовчого відділення**

**(англ. мовою)**

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