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HIGHER MATHEMATICS

Guidelines to independent work on the topic "Integral Calculus"

for Bachelor's (first) degree students of subject area 12 "Information Technology"

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The sufficient theoretical material on the academic discipline and typical examples are presented to help students master the material on the topic "Integral Calculus" and apply the obtained knowledge to practice. A detailed description and guidelines to perform tasks for independent work, references, a list of theoretical questions and a test for self-assessment are given in order to improve students' knowledge of the topic. The professional competences that students acquire as a result of studying the theoretical material and completing practical tasks on this topic are defined.

For Bachelor's (first) degree students of subject area 12 "Information Technology".

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Introduction

Integral is one of the basic concepts of mathematical analysis and mathematics in general. Integral calculus studies the properties and methods of computing indefinite and definite integrals. This branch of mathematics is of great practical importance and is widely used in various fields of human activity.

The purpose of the guidelines is to consider the main principles and concepts of integral calculus and to review some basic techniques for calculating indefinite integrals.

In the guidelines, first, the definitions of the antiderivative and indefinite integral are given, and the principal theorems and properties of the indefinite integral are formulated. The possibilities of the application of theorems in the calculation and verification of integrals are discussed.

The main methods of evaluation of integrals, such as integration by substitution or change of variables to find antiderivatives of complicated functions are given consideration. Two ways of applying the substitution rule are given.

An insight is taken into integration by parts with its advantages and application possibilities. The technique of using this method and typical cases of functions requiring integration of parts are provided.

In separate sections, the features of integration of rational, irrational and trigonometric functions are studied.

The consideration of each topic is accompanied by a sufficient number of examples. Finally, students are offered theoretical questions and tests for self-assessment.

1. The basic concepts of integral calculus

Integral is one of the fundamental concepts of mathematics.

Integration is the inverse process of differentiation. It means that, given a function $f(x)$, we wish to find a function $F(x)$ such that

$$F'(x) = f(x). \tag{1}$$

Definition 1. Any function $F(x)$ that satisfies the condition (1) is called an **antiderivative** of the function $f(x)$.

Theorem 1. If the function is continuous over an interval, then it has an antiderivative over this interval.

However, while we can find the derivative of any elementary function, the problem of finding the antiderivative is much more complicated.

A first observation is that the antiderivative, if it exists, is not unique. Indeed, suppose that the function $F(x)$ is an antiderivative of the function $f(x)$, so that $F'(x) = f(x)$. Then consider $G(x) = F(x) + C$, where C is any fixed real number. Then it is easy to see that $G'(x) = F'(x) + C' = f(x)$, so that $G(x)$ is also an antiderivative of $f(x)$.

Example 1. The functions $F_1(x) = \sin x$, $F_2(x) = \sin x + 5$, $F_3(x) = \sin x - \sqrt{2}$, as well as any function of the set $\{\sin x + C, \forall C \in \mathrm{R}\}$, are the antiderivatives of the function $f(x) = \cos x$, because

$$(\sin x)' = (\sin x + 5)' = (\sin x - \sqrt{2})' = (\sin x + C)' = \cos x.$$

A second observation, somewhat less obvious, is that for any given function $f(x)$, any two distinct antiderivatives of $f(x)$ must differ only by a constant. In other words, if $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, then $F(x) - G(x) = const$. It is summarized in the following statement.

Theorem 2. If $F(x)$ is an antiderivative of the function $f(x)$ over some interval I, then:

- for an arbitrary constant C the function $F(x) + C$ is also an antiderivative of $f(x)$ over this interval;
- any other antiderivative $G(x)$ of the function $f(x)$ over I is of the form $G(x) = F(x) + C$, where C is a certain real number.

Definition 2. The family $\{F(x) + C, \forall C \in \mathrm{R}\}$ of all antiderivatives of the function $f(x)$ is called an **indefinite integral** of this function and it is denoted by $\int f(x)dx$. We write

$$\int f(x)dx = F(x) + C. \qquad (2)$$

The symbol $\int$ is called an integral sign, the function $f(x)$ is called the integrand, x is the variable of integration.

Definition 3. The operation of finding the indefinite integral is called **integration** or **integrating**.

If we have to be specific about the integration variable, we say that we are integrating **with respect to** x.

Notice that both the integral sign $\int$ and dx are required. We can think of them as a set of parentheses. The dx that ends the integral is nothing more than the **differential.** The differential matches the variable of integration. (It becomes very important in using such integration techniques as substitution and in multivariable calculus where integrands may involve several variables.)

As the immediate consequence of formula (1) and our knowledge of derivatives we can easily obtain the antiderivatives of many elementary functions. Let us list the corresponding formulas (Table 1).

Table 1

The table of basic integrals

1	2
1	$\int 0dx = C$
2	$\int dx = x + C$
3	$\int x^{\alpha} dx = \dfrac{x^{\alpha+1}}{\alpha+1} + C, \quad \alpha \neq -1$
4	$\int \dfrac{dx}{x} = \int x^{-1} dx = \ln\lvert x\rvert + C$
5	$\int a^{x} dx = \dfrac{a^{x}}{\ln a} + C, \quad a > 0,\ a \neq 1$
6	$\int e^{x} dx = e^{x} + C$
7	$\int \sin x dx = -\cos x + C$
8	$\int \cos x dx = \sin x + C$
9	$\int \dfrac{dx}{\cos^{2} x} = \operatorname{tg} x + C$
10	$\int \dfrac{dx}{\sin^{2} x} = -\operatorname{ctg} x + C$

1	2
11	$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C, \quad a > 0$
12	$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C, \quad a > 0$
13	$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left\| \frac{x-a}{x+a} \right\| + C, \quad a \neq 0$
14	$\int \frac{dx}{\sqrt{x^2 + a}} = \ln \left\| x + \sqrt{x^2 + a} \right\| + C, \quad a \neq 0$
15	$\int \frac{dx}{x-a} = \ln \| x - a \| + C$
16	$\int \operatorname{tg} x dx = -\ln \|\cos x\| + C$
17	$\int \operatorname{ctg} x dx = \ln \|\sin x\| + C$
18	$\int \csc x dx = \int \frac{dx}{\sin x} = \ln \left\| \operatorname{tg} \frac{x}{2} \right\| + C$
19	$\int \sec x dx = \int \frac{dx}{\cos x} = \ln \left\| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right\| + C$

The formulae 15 – 19 are not really basic but it is useful to have them at our disposal.

Example 2. There are some examples of using the basic formulae given in Table 1:

a) $\int \sqrt{x} dx = \int x^{\frac{1}{2}} dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}\sqrt{x^3} + C$ (formula 3, $\alpha = \frac{1}{2}$);

b) $\int \frac{dx}{\sqrt[3]{x^2}} = \int x^{-\frac{2}{3}} dx = \frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} + C = \frac{x^{\frac{1}{3}}}{\frac{1}{3}} + C = 3 \cdot \sqrt[3]{x} + C$ (formula 3, $\alpha = -\frac{2}{3}$);

c) $\int \frac{dx}{\sqrt{5 - x^2}} = \arcsin \frac{x}{\sqrt{5}} + C$ (formula 11, $a = \sqrt{5}$);

d) $\int \frac{dx}{\sqrt{x^2-5}} = \ln = \ln\left|x+\sqrt{x^2-5}\right|$ (formula 14, $a=-5$).

Don't forget "$+C$" at the end, it is important.

2. The properties of the indefinite integral

It is obvious that many characteristic properties of the indefinite integral can be obtained simply by referring to various rules concerning derivatives. These properties occur very useful in calculating and verifying the integrals of more complicated functions, so we list here a number of such results.

Suppose that functions $f(x)$ and $g(x)$ have antiderivatives $F(x)$ and $G(x)$ respectively. Then the following statements hold.

1. A derivative of an indefinite integral equals the integrand:

$$\left(\int f(x)dx\right)' = f(x). \qquad (3)$$

This property can be used to verify the correctness of integration.

2. The indefinite integral of a derivative of a function equals the sum of this function and an arbitrary constant:

$$\int F'(x)dx = F(x)+C.$$

The table of basic indefinite integrals is the consequence of this property and the derivatives of elementary functions.

3. A differential of an indefinite integral equals an integrand expression:

$$d\int f(x)dx = f(x)dx.$$

4. The indefinite integral of a differential of a function equals the sum of this function and an arbitrary constant:

$$\int dF(x)dx = F(x)+C.$$

5. The constant multiple rule. Any fixed real factor can be taken outside the sign of the integral:

$$\int kf(x)dx = k\int f(x)dx = kF(x)+C, \quad k=const. \qquad (4)$$

6. The sum rule. The indefinite integral of an algebraic sum of functions that have antiderivatives equals an algebraic sum of the indefinite integrals of these functions:

$$\int (f(x)\pm g(x))dx = \int f(x)dx \pm \int g(x)dx = F(x)\pm G(x)+C. \qquad (5)$$

This property is valid for any finite number of addends.

Formulae (4), (5) can be easily verified by differentiation according to property **1** and due to derivative rules.

7. The invariance of the integration formula. If $\int f(x)dx = F(x) + C$ and $u = \varphi(x)$ is a continuously differentiable function over some interval, then

$$\int f(u)du = F(u) + C. \tag{6}$$

Indeed, due to the invariance of the form of the first differential we have

$$dF(u) = F'(u)du = f(u)du,$$

that implies

$$\int f(u)du = \int dF(u) = F(u) + C.$$

The integration regarding the table and the basic rules of indefinite integrals is called the **direct integration.**

Remember that we can always check the answer by differentiating and making sure we get the integrand.

Example 3. Evaluate the integrals by direct integration and check the results by differentiation:

a) $\int \left(1 + 2x - \frac{3}{x}\right)dx$; b) $\int \left(7 + 6t^2 - \frac{3}{t^2} - 2^t\right)dt$;

c) $\int \left(\frac{1}{\sqrt{4 - x^2}} + \frac{2}{4 + x^2} + \frac{3}{\sqrt{x^2 + 4}} - \frac{4}{4 - x^2}\right)dx$;

d) $\int \left(\frac{(x-3)^2}{\sqrt{x}} - \frac{2x^2}{x^2 + 1} + \mathrm{tg}^2 x\right)dx$.

Solution.

a) $\int \left(1 + 2x - \frac{3}{x}\right)dx = \left| \text{sum rule (5)} \right| =$

$$= \int dx + \int 2x dx - \int \frac{3}{x} dx = \left| \text{constant multiple rule (4)} \right| =$$

$$= \int dx + 2 \cdot \int x^1 dx - 3 \cdot \int \frac{dx}{x} = \left| \begin{array}{l} \text{formula 2;} \\ \text{formula 3 with } \alpha = 1; \\ \text{formula 4} \end{array} \right| =$$

$$= x + 2 \cdot \frac{x^{1+1}}{1+1} - 3 \cdot \ln|x| + C = x + x^2 - 3\ln|x| + C.$$

Here and below don't forget the absolute value bars in the argument of the logarithm function.

Taking into account that

$$(\ln|x|)' = \begin{bmatrix} \dfrac{1}{x} & \text{for } x > 0, \\ \dfrac{1}{-x} \cdot (-1) & \text{for } x > 0 \end{bmatrix} = \frac{1}{x}, \qquad (7)$$

we ensure that the derivative of the indefinite integral obtained coincides with the integrand:

$$\left(x + x^2 - 3\ln|x| + C\right)' = 1 + 2 \cdot x^{2-1} - 3 \cdot \frac{1}{x} + 0 = 1 + 2x - \frac{3}{x}.$$

b) Changing the integration variable in the integral just changes the variable in the answer:

$$\int \left(7 + 6t^2 - \frac{3}{t^2} - 2^t\right) dt = \left| \text{sum rule } (5) \right| =$$

$$= \int 7\,dt + \int 6t^2 dt - \int \frac{3}{t^2}\,dt - \int 2^t\,dt = \left| \text{constant multiple rule } (4) \right| =$$

$$= 7\int dt + 6\int t^2 dt - 3\int \frac{dt}{t^2} - \int 2^t\,dt = \begin{vmatrix} \text{formula } 2; \\ \text{formula } 3 \text{ with } \alpha = 2, \alpha = -2; \\ \text{formula } 5 \text{ with } a = 2 \end{vmatrix} =$$

$$= 7t + 6 \cdot \frac{t^{2+1}}{2+1} - 3 \cdot \frac{t^{-2+1}}{-2+1} - \frac{2^t}{\ln 2} + C =$$

$$= 7t + 2t^3 + \frac{3}{t} - \frac{2^t}{\ln 2} + C.$$

Differentiation with respect to t confirms the integration result:

$$\left(7t + 2t^3 + \frac{3}{t} - \frac{2^t}{\ln 2} + C\right)' =$$

$$= 7 + 2 \cdot 3t^{3-1} + 3 \cdot (-1) \cdot t^{-1-1} - \frac{1}{\ln 2} \cdot 2^t \ln 2 + 0 =$$

$$= 7 + 6t^2 - \frac{3}{t^2} - 2^t .$$

c) $\int \left(\frac{1}{\sqrt{4-x^2}} + \frac{2}{4+x^2} + \frac{3}{\sqrt{x^2+4}} - \frac{4}{4-x^2} \right) dx = \left| \text{sum rule (5)} \right| =$

$$= \int \frac{dx}{\sqrt{4-x^2}} + \int \frac{2dx}{4+x^2} + \int \frac{3dx}{\sqrt{x^2+4}} - \int \frac{4dx}{4-x^2} = \left| \text{constant multiple rule (4)} \right| =$$

$$= \int \frac{dx}{\sqrt{4-x^2}} + 2 \cdot \int \frac{dx}{x^2+4} + 3 \cdot \int \frac{dx}{\sqrt{x^2+4}} - (-4) \cdot \int \frac{dx}{x^2-4} =$$

$$= \left| \begin{matrix} \text{formula 11 with } a = 2; \\ \text{formula 12 with } a = 2; \\ \text{formula 13 with } a = 4; \\ \text{formula 14 with } a = 2 \end{matrix} \right| =$$

$$= \arcsin \frac{x}{2} + 2 \cdot \frac{1}{2} \operatorname{arctg} \frac{x}{2} + 3 \cdot \ln \left| x + \sqrt{x^2+4} \right| + 4 \cdot \frac{1}{2 \cdot 2} \ln \left| \frac{x-2}{x+2} \right| + C =$$

$$= \arcsin \frac{x}{2} + \operatorname{arctg} \frac{x}{2} + 3 \ln \left| x + \sqrt{x^2+4} \right| + \ln \left| \frac{x-2}{x+2} \right| + C .$$

The derivatives of each term are obtained with the help of the **chain rule**

$$\left(f\left(g(x)\right)\right)' = f'\left(g(x)\right) \cdot g'(x) \qquad (8)$$

and formula (7):

$$\left(\arcsin \frac{x}{2} \right)' = \frac{1}{\sqrt{1 - \left(\frac{x}{2}\right)^2}} \cdot \left(\frac{x}{2} \right)' = \frac{1}{\sqrt{\frac{4-x^2}{4}}} \cdot \frac{1}{2} = \frac{2}{\sqrt{4-x^2}} \cdot \frac{1}{2} = \frac{1}{\sqrt{4-x^2}} ;$$

$$\left(\operatorname{arctg} \frac{x}{2} \right)' = \frac{1}{1 + \left(\frac{x}{2}\right)^2} \cdot \left(\frac{x}{2} \right)' = \frac{2}{4+x^2} \cdot \frac{1}{2} = \frac{1}{4+x^2} ;$$

$$\left(\ln \left| x + \sqrt{x^2+4} \right| \right)' = \frac{1}{x + \sqrt{x^2+4}} \cdot \left(x + \sqrt{x^2+4} \right)' =$$

$$=\frac{1}{x+\sqrt{x^2+4}}\cdot\left(1+\frac{x}{\sqrt{x^2+4}}\right)=\frac{1}{x+\sqrt{x^2+4}}\cdot\frac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}}=\frac{1}{\sqrt{x^2+4}};$$

$$\left(\ln\left|\frac{x-2}{x+2}\right|\right)'=\frac{x+2}{x-2}\cdot\left(\frac{x-2}{x+2}\right)'=\left|\left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}\right|=\frac{x+2}{x-2}\cdot\frac{x+2-(x-2)}{(x+2)^2}=$$

$$=\frac{4}{(x-2)(x+2)}=\frac{4}{x^2-4}=-\frac{4}{4-x^2}.$$

This yields the correctness of the calculations in this case.

d) Sometimes evaluation can easily be reduced to direct integration by identity transformations:

$$\int\left(\frac{(x-3)^2}{\sqrt{x}}-\frac{2x^2}{x^2+1}+\mathrm{tg}^2x\right)dx=\left|\mathrm{tg}^2x=\frac{1}{\cos^2 x}-1\right|=$$

$$=\int\left(\frac{x^2-6x+9}{\sqrt{x}}-2\cdot\frac{x^2+1-1}{x^2+1}+\frac{1}{\cos^2 x}-1\right)dx=$$

$$=\int\left(x\sqrt{x}-6\sqrt{x}+\frac{9}{\sqrt{x}}-2\cdot\frac{x^2+1}{x^2+1}+2\cdot\frac{1}{x^2+1}+\frac{1}{\cos^2 x}-1\right)dx=$$

$$=\int\left(x^{\frac{3}{2}}-6\cdot x^{\frac{1}{2}}+9\cdot x^{-\frac{1}{2}}-2+2\cdot\frac{1}{x^2+1}+\frac{1}{\cos^2 x}-1\right)dx=$$

$$=\int x^{\frac{3}{2}}dx-6\int x^{\frac{1}{2}}dx+9\int x^{-\frac{1}{2}}dx+2\int\frac{dx}{x^2+1}+\int\frac{dx}{\cos^2 x}-3\int dx=$$

$$=\left|\begin{array}{l}\text{formula 3 with } \alpha=\frac{3}{2},\ \frac{1}{2},-\frac{1}{2};\\ \text{formula 12 with } a=1;\ \text{formula 9; formula 2}\end{array}\right|=$$

$$=\frac{2x^{\frac{5}{2}}}{5}-6\cdot\frac{2x^{\frac{3}{2}}}{3}+9\cdot 2x^{\frac{1}{2}}+2\mathrm{arctg}x+\mathrm{tg}x-3x+C=$$

$$=\frac{2}{5}x^2\sqrt{x}-4x\sqrt{x}+18\sqrt{x}-3x+2\mathrm{arctg}x+\mathrm{tg}x+C.$$

We suggest that you check this result yourself with the help of differentiation. Obviously, this will require transformations of the function you get.

It is worth to note that integration is more complicated than differentiation. As we have already mentioned the indefinite integral exists for any continuous, for example, any elementary function. Despite this, it is not always possible to express the antiderivative of a function by means of elementary functions. In particular, many integrals important for application define the special functions such as the Gaussian integral (Euler – Poisson integral), the error function, the sine and cosine integrals, elliptic integrals etc.

For example, no elementary function exists for such integrals:

$$\int e^{-x^2} dx \text{ – Poisson integral;}$$

$$\int \sin x^2 dx, \quad \int \cos x^2 dx \text{ – Fresnel integrals;}$$

$$\int \frac{\sin x}{x} dx \text{ – Sine integral;}$$

$$\int \frac{\cos x}{x} dx \text{ – Cosine integral;}$$

$$\int \frac{e^x}{x} dx, \quad \int \frac{e^{-x}}{x} dx \text{ – Exponential integral;}$$

$$\int \frac{dx}{\ln x} \text{ – integral Logarithm;}$$

$$\int x^{\alpha} \sin x dx, \quad \int x^{\alpha} \cos x dx, \quad \int x^{\alpha} e^x dx \qquad (\alpha \neq 0, 1, 2, \ldots) \text{ etc.}$$

3. The substitution rule for the indefinite integral

Here we discuss one of the main techniques of integration that are based on the **chain rule** (8) in differentiation. This method is called **integration by substitution** or **change of variables**. It allows us to find antiderivatives of more complicated functions.

The substitution rule. If $u = g(x)$ is a differentiable function on an interval $[a, b]$ and f is continuous on that interval, then

$$\int f(g(x)) g'(x) dx = \int f(u) du . \qquad (9)$$

There are two ways to apply this method. We can use formula (9) from left to right or from right to left. However our purpose is to transform the given integral into another that is easier to compute.

Notice that this technique doesn't always work. Besides, when it does, there may well be more than one suitable substitution.

Consider both versions of the change of variables and discuss the features of the use of this version.

Version 1. **Substitution** $u = g(x)$.

Let an integral $\int \varphi(x)dx$ be given and the antiderivative of the function $\varphi(x)$ is unknown. Suppose the integrand can be written in the form $\varphi(x) = f(g(x))g'(x)$.

The substitution $u = g(x)$ is made, then $du = g'(x)dx$, and we get

$$\int \varphi(x)dx = \int f(g(x))g'(x)dx = \left| \begin{array}{l} u = g(x) \\ du = g'(x)dx \end{array} \right| = \int f(u)du. \qquad (10)$$

It makes sense if we know the antiderivative of the function $f(u)$ that appears on the right-hand side of formula (10). Then the computing of the integral is as follows:

$$= \int f(u)du = F(u) + C = F(g(x)) + C.$$

We see that *in Version 1 the new variable* u *is written as a function of the old variable* x.

The first manner is also known as ***u*-substitution** as well.

Remember that upon performing the substitution every x in the integral (including dx) must disappear in the substitution process. And it's important that when calculating the integrals by substitution it is necessary to return to the original variable.

The natural question is how to identify the correct substitution. There is no general rule or prescription, it depends on the integral. Regarding formula (10) we should find a composite function $f(g(x))$ with a known antiderivative on the one hand, and the derivative $g'(x)$ of the new variable $u = g(x)$ on the other. So, each integral requires an individual analysis of the integrand to find a suitable changing of variables.

Example 4. Let's evaluate some integrals using the method of substitution:

a) $\int e^{\sin x}\cos x\,dx$; b) $\int x\sqrt{2-3x^2}\,dx$.

Solution.

a) We observe the composite function $e^{\sin x}$ (with the inside function $\sin x$) and the derivative of $\sin x$, so it's pretty clear that $\sin x$ is a new variable:

$$\int e^{\sin x}\cos x\,dx = \left| \begin{array}{l} u=\sin\ x \\ du=(\sin\ x)'dx=\cos x dx \end{array} \right| = \int e^u\,du =$$

$$= \left| \text{ formula } 6 \right| = e^u + C = e^{\sin x} + C.$$

b) Sometimes for using u-substitution only a constant factor is lacking. In our example, the first guess for the substitution is to make u be the stuff under the root. But at first we need to do some manipulation, namely, to multiply and divide the integrand by 6:

$$\int x\sqrt{2+3x^2}\,dx = \left| \begin{array}{l} u=2+3x^2 \\ du=(2+3x^2)'dx=6xdx \end{array} \right| =$$

$$= \frac{1}{6}\cdot\int 6x\sqrt{2+3x^2}\,dx = \frac{1}{6}\cdot\int\sqrt{u}\,du = \frac{1}{6}\cdot\int u^{\frac{1}{2}}\,du = \frac{1}{6}\cdot\frac{2}{3}\cdot u^{\frac{3}{2}} + C =$$

$$= \frac{1}{9}\sqrt{u^3} + C = \frac{1}{9}\sqrt{\left(2+3x^2\right)^3} + C.$$

It is useful to keep in mind the following **formula of linear substitution**:

$$\int f(kx+b)dx = \frac{1}{k}F(kx+b) + C, \qquad k,b = consts\ (k \neq 0), \qquad (11)$$

in particular,

$$\int f(kx)dx = \frac{1}{k}F(kx) + C, \qquad (k \neq 0).$$

Obviously, in formula (11) we have, in fact, the substitution

$$u = kx+b; \qquad du = (kx+b)'dx = kdx \quad \Leftrightarrow \quad \frac{du}{k} = dx.$$

Example 5. Let's apply the formula of linear substitution:

$$\int\left(\cos 2x + e^{-x} + \sqrt{5x-4}\right)dx = \left|\text{ sum rule (5) }\right| =$$

$$= \int \cos 2x dx + \int e^{-x} dx - \int (5x-4)^{\frac{1}{2}} dx = \left|\text{ formula of linear substituti on (11) }\right| =$$

$$= \frac{1}{2}\cdot \sin 2x + \frac{1}{-1}\cdot e^{-x} - \frac{1}{5}\cdot\frac{2}{3}(5x-4)^{\frac{3}{2}} + C = \frac{\sin 2x}{2} - e^{-x} - \frac{2}{15}\sqrt{(5x-4)^3} + C.$$

Version 2. Substitution $x = g(u)$**.**

Let an integral $\int f(x)dx$ be given and the antiderivative of the function $f(x)$ is unknown. If we make substitution $x = g(u)$, then $dx = g'(u)du$:

$$\int f(x)dx = \left| \begin{matrix} x = g(u) \\ dx = g'(u)du \end{matrix} \right| = \int f(g(u))g'(u)du. \qquad (12)$$

It makes sense if we know the antiderivative of the function $f(g(u))g'(u)$ that appears on the right-hand side of formula (12).

Notice that *in Version 2 the old variable* x *is written as a function of the new variable* u*.*

This version of substitution is mainly used for the transition from one class of functions to another.

Example 6. Evaluate the integrals of the irrational function by substitution:

a) $\int \frac{dx}{\sqrt{(1-x^2)^3}}$; b) $\int x\sqrt{1-x}dx$.

Solution.

a) This integral involving the root is reduced to the integral of the trigonometric function:

$$\int \frac{dx}{\sqrt{(1-x^2)^3}} = \left| \begin{matrix} x = \sin u \\ dx = (\sin u)' du = \cos u du \end{matrix} \right| = \int \frac{\cos u du}{\sqrt{(1-\sin^2 u)^3}} =$$

$$= \int \frac{\cos u du}{\left(\sqrt{\cos^2 u}\right)^3} = \int \frac{\cos u du}{\cos^3 u} = \int \frac{du}{\cos^2 u} = \text{tg} u + C =$$

$$= \frac{\sin u}{\cos u} + C = \frac{\sin u}{\sqrt{1-\sin^2 u}} + C = \frac{x}{\sqrt{1-x^2}} + C.$$

b) In this case we can transform the integrand to a polynomial by substitution:

$$\int x\sqrt{x+1}dx = \left| \begin{matrix} x = u^2 - 1 \\ dx = (u^2-1)'du = 2udu \end{matrix} \right| = \int \left(u^2 - 1\right) \cdot u \cdot 2udu =$$

$$= 2\int u^4 du - 2\int u^2 du = 2\cdot\frac{u^5}{5} - 2\cdot\frac{u^3}{3} + C = \frac{2}{5}\sqrt{(x+1)^5} - \frac{2}{3}\sqrt{(x+1)^3} + C.$$

There is another way of substitution that leads to direct integration (the table of basic integrals), namely, $x = v - 1$; $dx = dv$. We suggest that you do it yourself and compare the results.

Summarizing, *to apply the method of substitution in the indefinite integral* we have:

- to choose the substitution of one of the form $u = g(x)$ or $x = g(u)$;
- to compute $du = g'(x)dx$ or $dx = g'(u)du$;
- to change every expression with x in the original integral for u;
- to use the substitution rule formula (10) or (12).

At the end of this section let's make some remarks. If the substitution you made failed, it doesn't mean that this is a wrong method, it may well be an unsuitable substitution. Moreover, in case of success, another attempt may lead to a better change of variables.

Notice also that there is no simple routine that we can describe to help find a suitable substitution, even when the technique works. So, the only way to learn how to substitute is to just solve lots of problems of different kinds.

4. Integration by parts

At first, let us recall the **product rule** for differentiation:

$$(uv)' = u'v + uv', \qquad (13)$$

where $u(x)$ and $u(x)$ are differentiable functions.

Integrating (13) with respect to x we obtain:

$$\int (uv)'dx = \int u'vdx + \int uv'dx.$$

On the left-hand side we have the indefinite integral of uv and we can rewrite this equation as follows:

$$\int uv'dx = uv - \int u'vdx.$$

Finally, since $v'dx = dv$ and $u'dx = du$, we get the **integration by parts formula**:

$$\int udv = uv - \int vdu. \qquad (14)$$

Formula (14) is useful when the indefinite integral $\int vdu$ on the right-hand side is much easier to calculate than the original integral $\int udv$.

To apply the method of integration by parts to the indefinite integral we have:

- to identify u and dv;
- to compute $du = u'dx$ and $v = \int dv$;
- to use formula (14) of integration by parts.

Example 7. Let us integrate by parts the integral $\int (3x+1)\cdot e^x dx$. Indeed, choosing $u = 3x+1$ seems good, because after differentiation x will drop out, and the integral of the exponential function is one of the basic formulae:

$$\int (3x+1)\cdot e^x dx = \left| \begin{array}{ll} u = 3x+1, & du = u'dx = 3dx \\ dv = e^x dx, & v = \int dv = \int e^x dx = e^x \end{array} \right| =$$

$$= (3x+1)e^x - \int e^x \cdot 3dx = (3x+1)e^x - 3e^x + C = (3x-2)e^x + C.$$

The question is how to identify the need for integration by parts and then how to make a correct choice of u and dv.

The good news is that there are some typical cases (Table 2). Of course, they do not exhaust the possibility of applying this method.

Table 2

The types of functions integratable by parts

Type	Kind of integral	Factor u	Factor dv
1	2	3	4
The 1st type	$\int P_n(x)\cdot a^x dx*$	$u = P_n(x)$	$dv = a^x dx$
	$\int P_n(x)\cdot e^x dx$		$dv = e^x dx$
	$\int P_n(x)\cdot \sin xdx$		$dv = \sin xdx$
	$\int P_n(x)\cdot \cos xdx$		$dv = \cos xdx$

Table 2 (the end)

<table>
<tr><th>1</th><th>2</th><th>3</th><th>4</th></tr>
<tr><td rowspan="5">The 2nd type</td><td>$\int P_n(x)\cdot \log_a^k xdx$</td><td>$u=\log_a^k x$</td><td rowspan="5">$dv=P_n(x)dx$</td></tr>
<tr><td>$\int P_n(x)\cdot \arcsin xdx$</td><td>$u=\arcsin x$</td></tr>
<tr><td>$\int P_n(x)\cdot \arccos xdx$</td><td>$u=\arccos x$</td></tr>
<tr><td>$\int P_n(x)\cdot \mathrm{arctg} xdx$</td><td>$u=\mathrm{arctg} x$</td></tr>
<tr><td>$\int P_n(x)\cdot \mathrm{arcctg} xdx$</td><td>$u=\mathrm{arcctg} x$</td></tr>
<tr><td rowspan="2">The 3d type</td><td>$\int a^x\cdot \sin bxdx$</td><td>$u=\sin bx$
or $u=a^x$</td><td>$dv=a^x dx$
or $dv=\sin bxdx$</td></tr>
<tr><td>$\int a^x\cdot \cos bxdx$</td><td>$u=\cos bx$
or $u=a^x$</td><td>$dv=a^x dx$
or $dv=\cos bxdx$</td></tr>
<tr><td rowspan="3">The 4th type</td><td>$\int \sqrt{a^2-x^2}dx$</td><td>$u=\sqrt{a^2-x^2}$</td><td rowspan="3">$dv=dx$</td></tr>
<tr><td>$\int \sqrt{a^2+x^2}dx$</td><td>$u=\sqrt{a^2+x^2}$</td></tr>
<tr><td>$\int \sqrt{x^2-a^2}dx$</td><td>$u=\sqrt{x^2-a^2}$</td></tr>
</table>

* $P_n(x)=a_n x^n+\ldots+a_1 x+a_0$ is a polynomial.

Thus when faced with an integral, we must first realize the type of integral to correctly apply formula (14).

Example 8. Consider the examples of integration by parts:

a) $\int x\ln xdx$;   b) $\int e^x \sin 2xdx$.

a) We can quickly recognize the second type of those mentioned in Table 2:

$$\int x\ln xdx=\left|\begin{array}{ll} u=\ln x, & du=u'dx=\frac{1}{x}dx \\ dv=xdx, & v=\int dv=\int xdx=\frac{x^2}{2} \end{array}\right|=$$

$$=\frac{x^2}{2}\ln x-\frac{1}{2}\int x^2\cdot\frac{1}{x}dx=\frac{x^2}{2}\ln x-\frac{x^2}{4}+C.$$

b) This one is of the third type, because e^x is a particular case of a^x. Here, breaking up into u and dv is arbitrary, but integration is a little tricky.

We have to apply the integration-by-parts formula twice, and then express the given integral from the resulting equation:

$$I=\int e^{x}\sin 2x dx=\left|\begin{matrix} u=e^{x} & \Rightarrow & du=e^{x}dx \\ dv=\sin 2x dx & & v=-\dfrac{\cos 2x}{2}\end{matrix}\right|=$$

$$=e^{x}\cdot\left(-\frac{\cos 2x}{2}\right)-\int\left(-\frac{\cos 2x}{2}\right)\cdot e^{x}dx=-\frac{e^{x}\cos 2x}{2}+\frac{1}{2}\cdot\int e^{x}\cos 2x dx=$$

$$=\left|\begin{matrix} u=e^{x} & \Rightarrow & du=e^{x}dx \\ dv=\cos 2x dx & & v=\dfrac{\sin 2x}{2}\end{matrix}\right|=-\frac{e^{x}\cos 2x}{2}+\frac{1}{2}\left(e^{x}\frac{\sin 2x}{2}-\int\frac{\sin 2x}{2}e^{x}dx\right)=$$

$$=-\frac{e^{x}\cos 2x}{2}+\frac{1}{4}\cdot e^{x}\sin 2x-\frac{1}{4}\int e^{x}\sin 2x dx=-\frac{e^{x}\cos 2x}{2}+\frac{e^{x}\sin 2x}{4}-\frac{1}{4}\cdot I\,.$$

To find the integral we solve the linear equation on this integral:

$$I=-\frac{e^{x}\cos 2x}{2}+\frac{e^{x}\sin 2x}{4}-\frac{1}{4}I\,;$$

$$I+\frac{1}{4}I=-\frac{e^{x}\cos 2x}{2}+\frac{e^{x}\sin 2x}{4}\,;$$

$$\frac{5}{4}I=-\frac{e^{x}\cos 2x}{2}+\frac{e^{x}\sin 2x}{4}\,.$$

Finally,

$$I=\int e^{x}\sin 2x dx=\frac{e^{x}}{5}\left(\sin 2x-2\cos 2x\right)+C\,.$$

5. Integration of polynomial fractions

In this section we discuss the integration of rational expressions of polynomials.

We start with so called **elementary** algebraic fractions. They are of four types, namely:

1. $\dfrac{A}{x-a}$ **2.** $\dfrac{B}{\left(x-b\right)^{n}}$ **3.** $\dfrac{Ax+B}{x^{2}+px+q}$. **4.** $\dfrac{Ax+B}{(x^{2}+px+q)^{n}}$, where n is a natural number more than one; square trinomials have no real roots, i.e. $p^2 - 4q < 0$.

Let us take a look at the integrals of each type of elementary fractions.

The indefinite integrals of fractions of the first and second types are easily computed by linear substitution formula (11):

$$\textbf{1. } \int \frac{A}{x-a}\,dx = A\ln|x-a| + C. \qquad (15)$$

$$\textbf{2. } \int \frac{B}{(x-b)^n}\,dx = B\int (x-b)^{-n}\,dx = \frac{B}{1-n}\cdot\frac{1}{(x-b)^{n-1}} + C. \qquad (16)$$

For example,

$$\int \frac{5}{x-3}\,dx = 5\ln|x-3| + C, \qquad \int \frac{10}{(x+7)^6}\,dx = -\frac{2}{(x+7)^5} + C.$$

Similarly, regarding (11) we get indefinite integrals in more general cases:

$$\int \frac{A}{kx+b}\,dx = -\frac{A}{k}\cdot\ln|kx+b| + C; \qquad (17)$$

$$\int \frac{B}{(kx+b)^n}\,dx = -\frac{B}{k(n-1)}\cdot\frac{1}{(kx+b)^{n-1}} + C. \qquad (18)$$

3. $I = \int \frac{Ax+B}{x^2+px+q}\,dx$ is integrated by the substitution method.

When integrating the fraction of the third type, we have:

• to select the full square in the denominator:

$$x^2 + px + q = \left(x + \frac{p}{2}\right)^2 + q - \frac{p^2}{4};$$

• to enter a new variable $x + p/2 = t$:

$$I = \left| \begin{array}{c} x + p/2 = t \\ x = t - p/2 \\ dx = dt \end{array} \right| = \int \frac{A\left(t - \frac{p}{2}\right) + B}{t^2 + q - \frac{p^2}{4}}\,dt;$$

• to integrate two terms we obtained:

$$\int \frac{At}{t^2+q-\frac{p^2}{4}}dt+\int \frac{B-\frac{Ap}{2}}{t^2+q-\frac{p^2}{4}}dt.$$

Example 9. Compute the indefinite integral from the fraction of type 3.

$$\int \frac{3x-1}{x^2-4x+8}dx=\int \frac{3x-1}{(x-2)^2+4}\,dx=\left|\begin{matrix} x-2=t \\ x=t+2 \\ dx=dt \end{matrix}\right|=\int \frac{3(t+2)-1}{t^2+4}dt=$$

$$=3\int \frac{t}{t^2+4}\,dt+5\int \frac{dt}{t^2+4}=\frac{3}{2}\ln(t^2+4)+\frac{5}{2}\operatorname{arctg}\frac{t}{2}+C=$$

$$=\frac{3}{2}\ln(x^2-4x+8)+\frac{5}{2}\operatorname{arctg}\frac{x-2}{2}+C.$$

The considered approach can be applied to some integrals involving quadratic expressions (a square trinomial), namely:

1) $\int \frac{Ax+B}{x^2+px+q}dx$ – an indefinite integral in which the square trinomial has a positive discriminant $\left(p^2-4q>0\right)$;

2) $\int \frac{Ax+B}{ax^2+bx+c}dx$ – an indefinite integral involving the square trinomial with the leading coefficient;

3) $\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$ – an indefinite integral with a square root of a general quadratic expression in the denominator.

4. $I_n=\int \frac{Ax+B}{(x^2+px+q)^n}dx$ is reduced to an integral of type 3.

Integration of a type 4 fraction is generally very cumbersome: by means of multiple integration by parts it is reduced to an integral of a fraction of type 3.

A recurrent formula for $I_n = \int \frac{dx}{(x^2+a^2)^n}$ has the form:

$$I_{n+1} = \frac{1}{2an^2} \cdot \frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2na^2} I_n . \qquad (19)$$

Since we know $I_1 = \int \frac{dx}{x^2+a^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C$, then we can find the integral I_n for any n.

Now consider a rational expression in the form:

$$R(x) = \frac{P_n(x)}{Q_m(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0}, \qquad (20)$$

where $P_n(x)$, and $Q_m(x)$ are polynomials of degrees $n \in \mathbf{N} \bigcup \{0\}$ and $m \in \mathbf{N}$ respectively.

Definition 4. The rational fraction (20) is called **proper** if the degree of the numerator $P_n(x)$ is less than the degree of the denominator $Q_m(x)$, that is $n < m$; otherwise, if $n \geq m$, the fraction is called **improper.**

To integrate the polynomial fraction (20) we must first perform **partial fraction decomposition** (**partial fraction expansion)**, that is express the fraction as a sum of a polynomial (possibly zero) and one or several fractions with a simpler denominator.

An improper fraction can always be presented as the sum of a polynomial $T_{n-m}(x)$ and fractions:

$$R(x) = \frac{P_n(x)}{Q_m(x)} = T_{n-m}(x) + \frac{P_{n_1}(x)}{Q_m(x)}, \qquad (21)$$

where $T_{n-m}(x)$ and $P_{n_1}(x)$ are polynomials, $0 \leq n_1 \leq m-1$.

Example 10. Let us present the polynomial fraction

$$R(x)=\frac{2x^3+1}{x^2-x+2}$$

in the form (21).

We have:

$$\begin{array}{rrr|l}
2x^3 & & +1 & \\
-\;2x^3-2x^2 & +4x & & x^2-x+2 \\ \cline{1-3} \cline{4-4}
2x^2 & -4x & +1 & 2x+2 \\
-\;2x^2 & -2x & +4 & \\ \cline{1-3}
 & -2x & -3 &
\end{array}.$$

Then

$$R(x)=\frac{2x^3+1}{x^2-x+2}=2x+2+\frac{-2x-3}{x^2-x+2}.$$

Theorem 3. Every proper rational fraction can be decomposed into a sum of elementary fractions.

Remember that partial fractions can only be decomposed if the degree of the numerator is strictly less than the degree of the denominator.

To decompose a proper rational fraction $R(x)=\dfrac{P_n(x)}{Q_m(x)}$ $(n<m)$ into a sum of elementary fractions we have:

- to factorize the denominator $Q_m(x)$ (present it as a product of linear factors $(x-b)$ and quadratic factors (x^2+cx+d) with negative discriminators);
- to assign the simplest rational algebraic fraction or amount to each factor of the denominator (Table 3);
- to write $R(x)$ as a sum of all elementary fractions with unknown numerators (we assign variables, usually capital letters, to these unknown values);
- to determine the coefficients of partial fraction expansion.

Table 3

The correspondence between the factors of the denominator and elementary fractions

No.	The factor in the denominator	The term in the partial fraction decomposition
1	$x-a$	$\frac{A}{x-a}$
2	$(x-b)^l$	$\frac{B_1}{x-b}+\frac{B_2}{(x-b)^2}+\ldots+\frac{B_l}{(x-b)^l}$
3	x^2+px+q	$\frac{Cx+D}{x^2+px+q}$
4	$\left(x^2+px+q\right)^s$	$\frac{C_1x+D_1}{x^2+px+q}+\frac{C_2x+D_2}{\left(x^2+px+q\right)^2}+\ldots+\frac{C_sx+D_s}{\left(x^2+px+q\right)^s}$

To find the expansion coefficients we can use **the method of indefinite coefficients** according to which the finding of the expansion coefficients is reduced to the solution of the system of linear equations. The method is based on the following properties:

a) the equality is not broken if both of its parts are multiplied by the same expression;

b) equal polynomials have equal coefficients of corresponding powers of the independent variable.

Example 11. Consider examples of the integration of rational algebraic fractions: a) $\int \frac{x^2dx}{x^3-x^2+x-1}$; b) $\int \frac{dx}{x(x-13)^2}$.

Solution.

a) $\int \frac{3x^2+x-2}{x^3-x^2+x-1}dx$.

Since the second degree of the polynomial in the numerator is smaller than the third degree of the denominator, we have a proper rational fraction.

The first thing is to factorize the denominator:

$$x^3-x^2+x-1=(x-1)\left(x^2+1\right).$$

So, $\frac{3x^2+x-2}{x^3-x^2+x-1}=\frac{3x^2+x-2}{(x-1)\left(x^2+1\right)}$.

Now we get the form of the partial fraction decomposition:

$$\frac{3x^2+x-2}{(x-1)(x^2+1)}=\frac{A}{(x-1)}+\frac{Bx+C}{(x^2+1)}.$$

We multiply both parts of the equality by the denominator and thus get the equation:

$$3x^2+x-2=A(x^2+1)+(Bx+C)(x-1);$$

$$3x^2+x-2=x^2(A+B)+x(C-B)+A-C.$$

Let's find the expansion coefficients A, B, C. It's useful to begin with the substitution of the root $x=1$ of the denominator:

$$3+1-2=A(1+1)\quad\Rightarrow A=1.$$

Then we solve the linear system:

$$\begin{aligned} x^2:&\quad 3=A+B,\\ x:&\quad 1=C-B,\\ x^0:&\quad -2=A-C,\end{aligned}$$

$$\Rightarrow B=3-A=3-1=2;\qquad C=1+B=1+2.$$

So,

$$\frac{3x^2+x-2}{(x-1)(x^2+1)}=\frac{1}{(x-1)}+\frac{2x+3}{(x^2+1)};$$

$$\int\frac{3x^2+x-2}{(x-1)(x^2+1)}dx=\int\frac{dx}{x-1}+\int\frac{2x+3}{(x^2+1)}dx=$$

$$=\int\frac{dx}{x-1}+\int\frac{2xdx}{(x^2+1)}+\int\frac{3dx}{(x^2+1)}=\int\frac{dx}{x-1}+\int\frac{d(x^2+1)}{(x^2+1)}+3\cdot\int\frac{dx}{(x^2+1)}=$$

$$=\ln|x-1|+2\ln|x^2+1|+3\operatorname{arctg}x+C.$$

b) $\int\frac{dx}{x(x-3)^2}$.

This time the denominator is already factorized, so let's proceed to the partial fraction decomposition:

$$\frac{1}{x(x-3)^2}=\frac{A}{x}+\frac{B}{x-3}+\frac{C}{(x-3)^2}.$$

Multiplying by the denominator we get:

$$1 = A(x-3)^2 + Bx(x-3) + Cx;$$

$$1 = x^2(A+B) + x(6A - 3B + C) + 9A.$$

At first we substitute the roots $x_1 = 0,\ x_2 = 3$ of the denominator:

$$x_1 = 0 \quad \Rightarrow 1 = 9A \quad \Rightarrow A = \frac{1}{9};$$

$$x_2 = 3 \quad \Rightarrow 1 = 3C \quad \Rightarrow C = \frac{1}{3}.$$

Then

$$\begin{aligned} x^2: &\quad 0 = A + B &&\Rightarrow B = -A = -\frac{1}{9}, \\ x: &\quad 0 = 6A - 3B + C, && \\ x^0: &\quad 1 = 9A && \end{aligned}$$

and the result of decomposition is as follows:

$$\frac{1}{x(x-3)^2} = \frac{1}{9}\cdot\frac{1}{x} - \frac{1}{9}\cdot\frac{1}{x-3} + \frac{1}{3}\cdot\frac{1}{(x-3)^2}.$$

$$\int \frac{dx}{x(x-3)^2} = \int\left(\frac{1}{9}\cdot\frac{1}{x} - \frac{1}{9}\cdot\frac{1}{x-3} + \frac{1}{3}\cdot\frac{1}{(x-3)^2}\right)dx =$$

$$= \frac{1}{9}\cdot\int\frac{dx}{x} - \frac{1}{9}\cdot\int\frac{dx}{x-3} + \frac{1}{3}\cdot\int\frac{dx}{(x-3)^2} =$$

$$= \frac{1}{9}\ln|x| - \frac{1}{9}\ln|x-3| - \frac{1}{3(x-3)} + C.$$

6. Integration of trigonometric functions

We begin with the technique that allows us to reduce the integration of trigonometric functions to the integration of rational algebraic fractions. In this context we consider several types of integrals.

Denote $R\big(u(x), v(x)\big)$ a rational function of its arguments.

Integrals of the type $\int R(\sin x, \cos x)dx$ can always be computed using the substitution $t = \operatorname{tg}\frac{x}{2}$. Then

$$\frac{x}{2} = \operatorname{arctg} t \;\Rightarrow\; x = 2\operatorname{arctg} t \;\Rightarrow\; dx = \frac{2dt}{1+t^2},$$

and

$$\sin x = \frac{2\operatorname{tg}(x/2)}{1+\operatorname{tg}^2(x/2)} = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-\operatorname{tg}^2(x/2)}{1+\operatorname{tg}^2(x/2)} = \frac{1-t^2}{1+t^2}.$$

Since the functions

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2} \quad \text{and} \quad dx = \frac{2dt}{1+t^2} \tag{22}$$

are rational functions of the variable x, the given integral turns into the integral of a polynomial fraction with respect to the variable t (the results of arithmetic operations over the fractions are fractions):

$$I = \int R(\sin x, \cos x)dx = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \cdot \frac{2dt}{1+t^2} = \int r(t)dt.$$

Example 12. Let's evaluate the integral $\int \frac{dx}{3-2\cos x}$.

By substitution $t = \operatorname{tg}\frac{x}{2}$ and using formula (22) we obtain:

$$\int \frac{dx}{3-2\cos x} = \left| \begin{array}{l} t = \operatorname{tg}\frac{x}{2} \\ dx = \frac{2dt}{1+t^2} \\ \cos x = \frac{1-t^2}{1+t^2} \end{array} \right| = \int \frac{1}{3 - 2\cdot\frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2} =$$

$$= 2\int \frac{dt}{\left(3+3t^2-2(1-t^2)\right)} = 2\int \frac{dt}{\left(1+5t^2\right)} = \frac{2}{5}\int \frac{dt}{\left(\frac{1}{5}+t^2\right)} =$$

$$= \left| \text{formula } 11 \text{ with } a = \frac{1}{\sqrt{5}} \right| = \frac{2}{5}\sqrt{5}\cdot arctg\left(t\sqrt{5}\right) = \frac{2}{\sqrt{5}} \operatorname{arctg}\left(\sqrt{5}\operatorname{tg}\frac{x}{2}\right) + C.$$

The integration of $R(\sin x, \cos x)$ by the substitution $t = \operatorname{tg}\frac{x}{2}$ is sure to give the desired result, but it is because of its generality that this method may not be the best from the point of view of brevity and simplicity of the transformation involved.

There are some recommendations that are useful for special cases of the integrand.

1. If $R(\sin x, \cos x)$ is an odd function with respect to $\sin x$, i.e.

$$R(-\sin x, \cos x) = -R(\sin x, \cos x),$$

we use the substitution $t = \cos x$.

2. If $R(\sin x, \cos x)$ is an odd function with respect to $\cos x$, i.e.

$$R(\sin x, -\cos x) = -R(\sin x, \cos x),$$

we use the substitution $t = \sin x$.

3. If $R(\sin x, \cos x)$ is an even function with respect to both arguments, i.e.

$$R(-\sin x, -\cos x) = R(\sin x, \cos x),$$

we use the substitution $t = \operatorname{tg} x$ and the formulae

$$\sin^2 x = \frac{\operatorname{tg}^2 x}{1 + \operatorname{tg}^2 x} = \frac{t^2}{1 + t^2}, \quad \cos^2 x = \frac{1}{1 + \operatorname{tg}^2 x} = \frac{1}{1 + t^2}$$

or the substitution $t = \operatorname{ctg} x$ and the formulae

$$\sin^2 x = \frac{1}{1 + \operatorname{ctg}^2 x} = \frac{1}{1 + t^2}, \quad \cos^2 x = \frac{\operatorname{ctg}^2 x}{1 + \operatorname{ctg}^2 x} = \frac{t^2}{1 + t^2}.$$

4. Integrals of the type $\int R(\sin x) \cos x dx$ are integrated using the substitution $t = \sin x$.

5. Integrals of the type $\int R(\cos x) \sin x dx$ are integrated using the substitution $t = \cos x$.

6. Integrals of the type $\int R(\operatorname{tg} x) dx$ are integrated using the substitution $t = \operatorname{tg} x$.

Now consider a few special cases widely used in practice.

The integral of the form

$$I = \int \sin^m x \cdot \cos^n x \, dx, \ m \in \mathbf{Z}, \ n \in \mathbf{Z}, \text{ except } m = n = 0$$

is a special case of 1−3, so:

- if m is odd, we put $t=\cos x$;
- if n is odd, we put $t=\sin x$;
- if m and n are even, we put $t=\mathrm{tg}x$ or $t=\mathrm{ctg}x$.

In the third case, instead of the proposed substitutions, formulae for decreasing powers are often used (trigonometric functions):

$$\sin^2 x=\frac{1-\cos 2x}{2},\quad \cos^2 x=\frac{1+\cos 2x}{2},$$

$$\sin x\cos x=\frac{\sin 2x}{2}.$$

Example 13. Let us find $\int\frac{\cos^5 x}{\sin^2 x}dx$.

We have $m=-2,\ n=5$. Since the integrand contains an odd degree of $\cos x$, we perform substitution $t=\sin x$, write $\cos^5 x$ as $t=\cos x\cos^4 x$ and apply the identity $\sin^2 x+\cos^2 x=1$:

$$\int\frac{\cos^5 x}{\sin^2 x}dx=\left|\begin{array}{c} t=\sin x,\ dt=\cos x\,dx \\ \cos^2 x=1-\sin^2 x=1-t^2\end{array}\right|=$$

$$=\int\frac{\cos^4 x\cos x}{\sin^2 x}\,dx=\int\frac{(1-t^2)^2}{t^2}dt=\int\frac{1-2t^2+t^4}{t^2}dt=$$

$$=\int(t^{-2}-2+t^2)\,dt=\frac{t^{-1}}{-1}-2t+\frac{t^3}{3}+C=\frac{-1}{\sin x}-2\sin x+\sin^2 x+C.$$

Now consider the integrals of the product of the first degrees of sines and cosines with different arguments:

$$\int\sin mx\cdot\cos nx\,dx,\quad \int\cos mx\cdot\cos nx\,dx,\quad \int\sin mx\cdot\sin nx\,dx,$$

where m and n are real numbers.

Calculation of these indefinite integrals is reduced to direct integration through the use of trigonometric formulae:

$$\sin\alpha\cdot\sin\beta=\frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha-\beta)\right),$$

$$\cos\alpha\cdot\cos\beta=\frac{1}{2}\left(\cos(\alpha-\beta)+\cos(\alpha+\beta)\right),$$

$$\sin\alpha\cdot\cos\beta=\frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right).$$

Example 14. Let us find the integral $\int\sin 3x\cdot\cos 7x\,dx$.

Using the formula above we get:

$$\int\sin 3x\cdot\cos 7x\,dx=\left|\alpha=3x,\ \beta=7x\right|=\frac{1}{2}\int\left(\sin 10x+\sin(-4x)\right)dx=$$

$$=\left|\begin{array}{l}\sin(-4x)=-\sin 4x\\ \text{linear substitution formula}\end{array}\right|=$$

$$=\frac{1}{2}\left(\int\sin 10x\,dx-\int\sin 4x\,dx\right)=\frac{1}{8}\cos 4x-\frac{1}{20}\cos 10x+C.$$

At last, consider the integrals of natural degrees of the tangent and cotangent:

$$\int\operatorname{tg}^n x\,dx,\ \int\operatorname{ctg}^n x\,dx,\quad \text{where}\quad 2\le n\in\mathbf{N}.$$

In this case we use the substitutions $t=\operatorname{tg}x$ and $t=\operatorname{ctg}x$ respectively. Thus we have:

$$\int\operatorname{tg}^n x\,dx=\left|\begin{array}{c}t=\operatorname{tg}x,\ x=\operatorname{arctg}t\\ dx=\dfrac{1}{1+t^2}dt\end{array}\right|=\int\frac{t^n}{t^2+1}dt=\int r(t)\,dt;$$

$$\int\operatorname{ctg}^n x\,dx=\left|\begin{array}{c}t=\operatorname{ctg}x,\ x=\operatorname{arcctg}t\\ dx=-\dfrac{1}{1+t^2}dt\end{array}\right|=-\int\frac{t^n}{t^2+1}dt=-\int r(t)\,dt.$$

Example 15. Let's find the integral $\int\operatorname{ctg}^6 x\,dx$.

Let $t=\operatorname{ctg}x$, then:

$$\int\operatorname{ctg}^6 x\,dx=\left|\begin{array}{c}t=\operatorname{ctg}x,\ x=\operatorname{arcctg}t\\ dx=-\dfrac{dt}{1+t^2}\end{array}\right|=-\int\frac{t^6}{t^2+1}dt=-\int\left(t^4-t^2+1-\frac{1}{t^2+1}\right)dt=$$

$$=-\left(\frac{t^5}{5}-\frac{t^3}{3}+t+\operatorname{arcctg} t\right)+C=-\left(\frac{\operatorname{ctg}^5 x}{5}-\frac{\operatorname{ctg}^3 x}{3}+\operatorname{ctg} x+x\right)+C.$$

7. Integration of some irrational functions

In this section we consider the integration techniques that can be useful for some types of integrals with irrational terms in them, i. e. involving roots.

1. Integrals of the type

$$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_1}{n_1}}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{\frac{m_s}{n_s}}\right) dx, \qquad m_i \in \mathbf{Z},\ n_i \in \mathbf{Z}$$

(where R is a rational function of its arguments) can be reduced to the integrals of rational fractions using the substitution

$$\frac{ax+b}{cx+d}=t^k,$$

where k is the least common denominator of the fractions $\frac{m_i}{n_i}$, $i=\overline{1,s}$.

In particular, to simplify the integrals of the form

$$\int R\left(x,\ (ax+b)^{\frac{m_1}{n_1}}, \ldots, (ax+b)^{\frac{m_s}{n_s}}\right) dx$$

we use the substitution $ax+b=t^k$.

Example 16. Let's find the indefinite integral $\int \frac{dx}{\sqrt[3]{(2x-1)^2}-\sqrt{2x-1}}$.

In this case $\frac{m_1}{n_1}=\frac{2}{3}, \frac{m_2}{n_2}=\frac{1}{2}$, then $k=6$ and we apply the substitution $2x-1=t^6$:

$$\int \frac{dx}{\sqrt[3]{(2x-1)^2}-\sqrt{2x-1}}=\left|\begin{array}{c} 2x-1=t^6,\ t=\sqrt[6]{2x-1} \\ 2dx=6t^5dt \\ dx=3t^5dt \end{array}\right|=$$

$$=3\int\frac{t^5dt}{t^4-t^3}=3\int\frac{t^2dt}{t-1}=3\int\frac{(t^2-1)+1}{t-1}dt=$$

$$=3\int\left(t+1+\frac{1}{t-1}\right)dt=3\left(\frac{t^2}{2}+t+\ln|t-1|\right)+C=\left|\,t=\sqrt[6]{2x-1}\,\right|=$$

$$=3\left(\sqrt[3]{2x-1}+2\sqrt[6]{2x-1}+2\ln\left|\sqrt[6]{2x-1}-1\right|\right)+C.$$

2. Integrals of the types

1) $\int R\left(x,\sqrt{a^2-x^2}\right)dx$,

2) $\int R\left(x,\sqrt{a^2+x^2}\right)dx$,

3) $\int R\left(x,\sqrt{x^2-a^2}\right)dx$

can be reduced to the integrals of rational fractions using the following trigonometric substitutions respectively:

1) $x=a\sin t$ or $x=a\cos t$,
2) $x=a\,\mathrm{tg}t$ or $x=a\,\mathrm{ctg}t$,
3) $x=\dfrac{a}{\sin t}$ or $x=\dfrac{a}{\cos t}$.

Example 17. Compute the indefinite integral $\int\dfrac{dx}{x\sqrt{x^2+9}}$.

The integrand contains the quadratic term $x^2+a^2=x^2+3^2$ under the square root. So, we put $x=3\,\mathrm{tg}t$:

$$\int\frac{dx}{x\sqrt{x^2+9}}=\left|\begin{array}{c} x=3\,\mathrm{tg}\,t,\ \ dx=\dfrac{3}{\cos^2 t}dt \\ \sqrt{9+x^2}=3\sqrt{1+\mathrm{tg}^2 t}=\dfrac{3}{\cos t}\end{array}\right|=\int\frac{1}{3\,\mathrm{tg}\,t}\cdot\frac{\cos t}{3}\cdot\frac{3}{\cos^2 t}dt=$$

$$=\frac{1}{3}\int\frac{1}{\sin t}dt=\frac{1}{3}\ln\left|\,\mathrm{tg}\frac{t}{2}\right|+C=\left|\,t=\mathrm{arctg}\frac{x}{3}\right|=\frac{1}{3}\ln\left|\,\mathrm{tg}\frac{\mathrm{arctg}(x/3)}{2}\right|+C.$$

To simplify the resulting expression let's use the trigonometric relationships:

$$\operatorname{tg}\frac{t}{2}=\frac{1-\cos t}{\sin t}=\frac{1}{\sin t}-\operatorname{ctg}t=\sqrt{1+\operatorname{ctg}^2 t}-\operatorname{ctg}t=$$

$$=\left|\operatorname{tg}t=\frac{x}{3},\quad \operatorname{ctg}t=\frac{3}{x}\right|=\sqrt{1+\frac{9}{x^2}}-\frac{3}{x}=\frac{\sqrt{9+x^2}-3}{x}.$$

Finally we get $\int\frac{dx}{x\sqrt{x^2+9}}=\frac{1}{3}\ln\left|\frac{\sqrt{x^2+9}-3}{x}\right|+C.$

Theoretical questions for self-assessment

1. What is called an antiderivative, an integrand expression, an integration variable? Give examples.

2. Give a definition of an indefinite integral.

3. Formulate the basic properties of the indefinite integral.

4. Write down the table of basic indefinite integrals.

5. Which operation is more difficult: differentiation or integration? Justify your answer.

6. What are the main methods of integration you know?

7. What is the method of direct integration?

8. What is the difference between the change of a variable and substitution in the indefinite integral? Give examples.

9. Give the formula of integration by parts.

10. When is integration by parts applied?

11. Give examples of integrals that "are not taken".

12. What functions are called rational? Give examples.

13. What rational fraction is called proper (improper)?

14. What integrals are called proper and improper? Give examples.

15. Give a general scheme of integration of rational functions.

16. What types of irrational and transcendental functions can you integrate?

17. What is universal trigonometric substitution? When and for what purpose is it used?

18. Describe the essence of the method of indeterminate coefficients.

Test for self-assessment

1. When is $F(x)$ the antiderivative of $f(x)$?

a) $f'(x) = F(x)$; b) $F'(x) = f(x)$;

c) $f(x) = F(x) + C$; d) $F(x) = -f(x)$.

2. Choose the wrong properties of the indefinite integral:

a) $\int dF(x) = F(x) + C$; b) $d\left(\int f(x)dx\right) = f(x) + C$;

c) $\int f'(x)dx = f(x) + C$; d) $\int f(x)g(x)dx = \int f(x)dx \cdot \int g(x)dx$.

3. The integration-by-parts formula has the form:

a) $\int udv = uv - \int vdx$; b) $\int udv = uv + \int vdu$;

c) $\int udv = uv - \int vdu$; d) $\int udv = uv - \int du$.

4. Choose the integrals that can't be found by direct integration:

a) $\int 3x^2 dx$; b) $\int x^2 \cos x dx$; c) $\int 3\cos x dx$; d) $\int (3 + x)dx$.

5. Choose the integrals that can be found by direct integration:

a) $\int x2^x dx$; b) $\int (x - \pi)dx$; c) $\int \left(x + 2^x\right)dx$; d) $\int x2^{x^2} dx$.

6. Choose the integrals that can be calculated by substitution:

a) $\int \arccos x dx$; b) $\int \cos(4 - 3x)dx$; c) $\int x \ln x dx$; d) $\int \frac{\ln x dx}{x}$.

7. Which of the following integrals can't be found by substitution:

a) $\int \mathrm{arctg} x dx$; b) $\int e^{\sin x} \cos x dx$; c) $\int e^{3x+5} dx$; d) $\int \frac{\mathrm{arctg} x dx}{1 + x^2}$?

8. For which of the following integrals do we use integration by parts:

a) $\int (x^4 + 3)dx$; b) $\int e^{2x+1} dx$; c) $\int x \log_2 x dx$; d) $\int dx$?

9. Choose the integrals that can be calculated by parts:

a) $\int \arcsin x dx$; b) $\int \frac{dx}{\sqrt[3]{x} + \sqrt{x}}$; c) $\int \frac{x^6 dx}{3x^2 + 2x - 5}$; d) $\int \sqrt{x^2 + 1}$.

10. Which of the following integrals can't be expressed by elementary functions:

a) $\int x^2 \sin xdx$; b) $\int x \sin x^2 dx$; c) $\int \sin x^2 dx$; d) $\int \sin^2 xdx$?

11. Choose the integrals that can't be expressed by elementary functions:

a) $\int e^{-x^2} dx$; b) $\int e^{-x} dx$; c) $\int xe^x dx$; d) $\int \frac{e^x}{x} dx$.

12. The antiderivative of $f(x) = \sin x$ is:

a) $\sin x + C$; b) $-\sin x + C$; c) $\cos x + C$; d) $-\cos x + C$.

13. What is $\int \frac{dx}{\sqrt{x^2 - 4}}$:

a) $\frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| + C$; b) $\frac{1}{2} arctg \frac{x}{2} + C$;

c) $\arcsin \frac{x}{2} + C$; d) $\ln \left| x + \sqrt{x^2 - 4} \right| + C$?

14. Calculate $\int 3\cos xdx$:

a) $3\sin x + C$; b) $\frac{1}{3} \sin x$; c) $-3\sin x + C$; d) $-\frac{1}{3} x + C$.

15. Find $\int 3x^2 dx$:

a) $x^2 + C$; b) $x^3 + C$; c) $\frac{1}{3} x^3 + C$; d) $6x + C$.

16. What is $\int \sec^2 xdx$:

a) $\frac{1}{3} \sec^3 x + C$; b) $tgx + C$; c) $tg^2 x + C$; d) $2\sec^2 xtgx + C$?

17. Compute the indefinite integral $\int \left(3x^2 \left(2x^3 - 1\right)\right) dx$:

a) $x^3 \left(x^2 - 1\right) + C$; b) $x^3 \left(x^3 - 1\right) + C$;

c) $x^3\left(2x^3-1\right)+C$;   d) $x^3\left(\frac{1}{2}x^4-1\right)+C$.

18. Use integration by substitution to calculate $\int\left(\frac{x^2}{x^3+1}\right)dx$:

a) $\ln|x^3+1|+C$;   b) $\frac{1}{2}\ln|x^3+1|+C$;

c) $\frac{1}{3}\ln|x^3+1|+C$;   d) $\frac{4x^3}{3x^4+12}+C$.

19. Compute $\int 5\sin^4 x\cos xdx$:

a) $\cos^5 x\sin x+C$;   b) $\frac{1}{5}\sin^5 x+C$;

c) $\sin^5 x\cos x+C$;   d) $\sin^5 x+C$.

20. Use integration by substitution to find $\int 2\sec^2 xtgxdx$:

a) tg^2x+C;   b) $2\sec x+C$;

c) $\sec xtgx+C$;   d) $\sec xtg^2x+C$.

21. Use integration by parts to calculate $\int x\sin xdx$:

a) $-x\cos x+\sin x+C$;   b) $-x\cos x-\sin x+C$;

c) $-x\cos x+C$;   d) $-x\sin x+\cos x+C$.

22. What is $\int x\ln xdx$:

a) $\frac{x^2}{2}\ln x+C$;   b) $\frac{x^2}{2}\ln x-\frac{x}{2}+C$;

c) $\frac{x^2}{2}\ln x+\frac{x^2}{4}+C$;   d) $\frac{x^2}{2}\ln x-\frac{x^2}{4}+C$?

23. Compute the indefinite integral $\int\left(x+2\right)e^x dx$:

a) $\frac{x+1}{x+1}e^{x+1}+C$;   b) xe^x+C;

c) $\left(x+1\right)e^x+C$;   d) $\left(x+2\right)e^x+C$.

Individual tasks

Task 1. Find the integrals by substitution.

Table 4

1	2	3
1	a) $\int \frac{dx}{(x-5)^2+9}$	b) $\int \frac{e^{2x}dx}{e^x-1}$
2	a) $\int \sqrt{6x-5}dx$	b) $\int \frac{dx}{\sqrt{\left(1-x^2\right)\arcsin x}}$
3	a) $\int \sin\left(7x-3\right)dx$	b) $\int \frac{xdx}{2+x^4}$
4	a) $\int \frac{dx}{\cos^2(7x-8)}$	b) $\int \left(e^x-5\right)^{11} e^x dx$
5	a) $\int 3^{2x+3}dx$	b) $\int \frac{\sin 2x}{\sqrt[3]{1+\cos^2 x}}dx$
6	a) $\int \cos\left(5x+3\right)dx$	b) $\int \frac{\arcsin x dx}{\sqrt{1-x^2}}$
7	a) $\int \frac{dx}{4x^2+25}$	b) $\int x^3\sqrt{3x^4-1}dx$
8	a) $\int \sqrt[3]{4+\frac{x}{3}}dx$	b) $\int \frac{dx}{x\sqrt{4-\ln^2 x}}$
9	a) $\int \frac{dx}{(5x-8)^{10}}$	b) $\int \frac{xdx}{4x^2+5}$
10	a) $\int \cos\left(7-x\right)dx$	b) $\int \frac{x^3dx}{\sqrt{9-x^8}}$
11	a) $\int \frac{dx}{\sin^2\left(3x-1\right)}$	b) $\int \frac{x^3dx}{2+x^4}$
12	a) $\int \frac{dx}{\sqrt{8x+1}}$	b) $\int \left(2^x-1\right)^7 2^x dx$
13	a) $\int \sqrt[4]{2+5x}\,dx$	b) $\int \frac{dx}{(1+x^2)arctg^3 x}$
14	a) $\int \frac{dx}{(5x-1)^3}$	b) $\int x^4 e^{x^5-1}dx$

Table 4 (the end)

1	2	3
15	a) $\int \sin(4x-1)dx$	b) $\int \frac{x^4 dx}{\sqrt{3+x^{10}}}$
16	a) $\int \frac{dx}{\sqrt[4]{2x-3}}$	b) $\int x^4 \cos x^5 dx$
17	a) $\int \frac{dx}{(x-2)^2+1}$	b) $\int \frac{x^3}{\cos^2 x^4} dx$
18	a) $\int (3x-7)^8 dx$	b) $\int \frac{xdx}{(2x^2+3)^4}$
19	a) $\int \sqrt[3]{3x-5}\, dx$	b) $\int \frac{\operatorname{arctg} x dx}{1+x^2}$
20	a) $\int \frac{dx}{(2x-1)^7}$	b) $\int xe^{x^2} dx$
21	a) $\int \frac{dx}{(3x+2)^2-4}$	b) $\int e^x \sqrt[3]{\frac{e^x-1}{3}} dx$
22	a) $\int \sqrt{1-\frac{4x}{5}}\, dx$	b) $\int \frac{\ln^3 x+3}{x \ln x} dx$
23	a) $\int \frac{dx}{(x-2)^2+4}$	b) $\int \frac{\sin 2x}{\sqrt[3]{\cos^2 2x}} dx$
24	a) $\int \sqrt{\frac{2x}{3}+7}\, dx$	b) $\int \frac{\arccos x dx}{\sqrt{1-x^2}}$
25	a) $\int \frac{dx}{\sqrt{1-9x^2}}$	b) $\int x^2 \cdot \sqrt[3]{2x^3-9} dx$
26	a) $\int \sqrt{11-\frac{x}{4}}\, dx$	b) $\int \frac{e^x}{\sqrt[4]{e^x+1}} dx$
27	a) $\int \frac{dx}{25x^2+4}$	b) $\int x\sqrt{1-5x^2} dx$
28	a) $\int \sqrt{2-x} dx$	b) $\int e^{4-5x^2} xdx$
29	a) $\int e^{3-2x} dx$	b) $\int \frac{x^2 dx}{\sqrt{x^6+7}}$
30	a) $\int \frac{dx}{(5x-1)^2+7}$	b) $\int \frac{\sin x dx}{\sqrt{5-2\cos x}}$

Task 2. Integrate by parts.

Table 5

1	$\int x^2 \sin x dx$	16	$\int \arccos 4x dx$
2	$\int x \,\mathrm{arctg}\, x dx$	17	$\int 3^x \cos x dx$
3	$\int x \ln(x-1) dx$	18	$\int x e^{-x} dx$
4	$\int x^2 \cos x dx$	19	$\int (2-3x) \ln 4x dx$
5	$\int e^{2x} \sin x dx$	20	$\int \sqrt{x^2+16}\, dx$
6	$\int (x+3) \cos x dx$	21	$\int (2-x) \ln x dx$
7	$\int x^2 e^x dx$	22	$\int e^x \cos 2x dx$
8	$\int \sqrt{25-x^2}\, dx$	23	$\int (\ln x)^2 dx$
9	$\int \cos(\ln x) dx$	24	a) $\int e^{\frac{x}{2}} \cos x dx$
10	$\int \frac{\arcsin x}{\sqrt{1+x}} dx$	25	a) $\int x \arcsin \frac{1}{x} dx$
11	$\int \sqrt{9+x^2}\, dx$	26	a) $\int x^4 \ln x dx$
12	$\int (x-1) e^{4x} dx$	27	a) $\int \sin(\ln x) dx$
13	$\int \frac{x dx}{\cos^2 x}$	28	a) $\int \ln(3x-1) dx$
14	$\int \sqrt{x^2-5}\, dx$	29	a) $\int \frac{\arcsin x}{\sqrt{1-x}} dx$
15	$\int (x-4) \sin x dx$	30	$\int 2^x \cos x dx$

Task 3. Find the integrals of the rational fraction.

Table 6

1	2	3
1	a) $\int \frac{x^3-1}{x^2+2x+17} dx$	b) $\int \frac{10+x}{(x-1)(x^2+4)} dx$
2	a) $\int \frac{x^3-3}{x^2+12x+37} dx$	b) $\int \frac{3x+1}{(x+2)(x^2+1)} dx$

Table 6 (continuation)

1	2	3
3	a); $\int \frac{x^2}{x^2+10x+29}dx$	b) $\int \frac{x-9}{(x-1)(x^2+5)}dx$
4	a) $\int \frac{4x^2+5}{x^2+10x+34}dx$	b) $\int \frac{x+4}{(x-2)(x^2+7)}dx$
5	a) $\int \frac{x^2+3}{x^2+8x+20}dx$	b) $\int \frac{1-x}{(3x-2)(x^2+1)}dx$
6	a) $\int \frac{x^3}{x^2+4x+13}dx$	b) $\int \frac{x-7}{(x-3)(x^2+4)}dx$
7	a) $\int \frac{x^2-6}{x^2+2x+2}dx$	b) $\int \frac{6x+5}{(x-1)(x^2+5)}dx$
8	a) $\int \frac{x^3}{x^2+8x+17}dx$	b) $\int \frac{x-5}{(3x-4)(x^2+8)}dx$
9	a) $\int \frac{x^2-3}{x^2+2x+10}dx$	b) $\int \frac{4x+1}{(x+2)(x^2+2)}dx$
10	a) $\int \frac{x^3}{x^2+6x+10}dx$	b) $\int \frac{9x-1}{(x+4)(x^2+6)}dx$
11	a) $\int \frac{2-x^3}{x^2+4x+8}dx$	b) $\int \frac{8-x}{(x-1)(x^2+2)}dx$
12	a) $\int \frac{x-x^2}{x^2+4x+8}dx$	b) $\int \frac{x+2}{(x-5)(x^2+3)}dx$
13	a) $\int \frac{3-x^2}{x^2+6x+13}dx$	b) $\int \frac{5x+6}{(x-2)(x^2+9)}dx$
14	a) $\int \frac{x^3+1}{x^2+10x+26}dx$	b) $\int \frac{1-x}{(x-5)(x^2+3)}dx$
15	a) $\int \frac{2-x^3}{x^2+6x+13}dx$	b) $\int \frac{x+1}{(x-2)(x^2+9)}dx$
16	a) $\int \frac{x^2}{x^2+6x+10}dx$	b) $\int \frac{4}{(x+1)(x^2+6)}dx$

Table 6 (the end)

1	2	3
17	a) $\int \frac{x^4}{x^2+12x+40}dx$	b) $\int \frac{5-x}{(x+1)(x^2+11)}dx$
18	a) $\int \frac{7x^3-1}{x^2+4x+5}dx$	b) $\int \frac{x+17}{(x-2)(x^2+3)}dx$
19	a) $\int \frac{x^3-5}{x^2+2x+10}dx$	b) $\int \frac{3x+7}{(x+2)(x^2+2)}dx$
20	a) $\int \frac{x^2+x}{x^2+4x+13}dx$	b) $\int \frac{2x-1}{(x-3)(x^2+4)}dx$
21	a) $\int \frac{x^2}{x^2+4x+5}dx$	b) $\int \frac{x+2}{(x-2)(x^2+3)}dx$
22	a) $\int \frac{x^3+1}{x^2+2x+2}dx$	b) $\int \frac{1}{(x-1)(x^2+5)}dx$
23	a) $\int \frac{x^2+10}{x^2+12x+40}dx$	b) $\int \frac{x-3}{(x+1)(x^2+9)}dx$
24	a) $\int \frac{x^3+9}{x^2+10x+34}dx$	b) $\int \frac{x+2}{(x-7)(x^2+7)}dx$
25	a) $\int \frac{x^3}{x^2+12x+37}dx$	b) $\int \frac{2x+8}{(2x+1)(x^2+1)}dx$
26	a) $\int \frac{x^3+7}{x^2+10x+29}dx$	b) $\int \frac{9x-2}{(x-6)(x^2+5)}dx$
27	a) $\int \frac{x^3}{x^2+2x+17}dx$	b) $\int \frac{2x-5}{(x-1)(x^2+4)}dx$
28	a) $\int \frac{2x^2+1}{x^2+10x+26}dx$	b) $\int \frac{8x+2}{(3x-5)(x^2+3)}dx$
29	a) $\int \frac{x^2+24}{x^2+8x+20}dx$	b) $\int \frac{7-x}{(3x-2)(x^2+1)}dx$
30	a) $\int \frac{2x^3-3}{x^2+8x+17}dx$	b) $\int \frac{5x+2}{(x-4)(x^2+8)}dx$

Task 4. Find the integrals of the trigonometric function.

Table 7

1	2	3
1	a) $\int \sin^3 x \cos x dx$	b) $\int \frac{\cos x dx}{2-3\sin x}$
2	a) $\int \sin^4 x dx$	b) $\int \frac{dx}{3-4\cos^2 x}$
3	a) $\int \cos x \cos 6x dx$	b) $\int \frac{\cos x dx}{3-4\cos x}$
4	a) $\int \mathrm{ctg}^2 x dx$	b) $\int \frac{dx}{2-5\cos^2 x}$
5	a) $\int \frac{\sin^5 x}{\cos^3 x} dx$	b) $\int \frac{dx}{3-8\cos x}$
6	a) $\int \sin^3 x \cos^4 x dx$	b) $\int \frac{dx}{6-\cos^2 x}$
7	a) $\int \sin^4 x dx$	b) $\int \frac{\sin x dx}{1-7\sin x}$
8	a) $\int \sin 8x \sin x dx$	b) $\int \frac{dx}{3-4\sin^2 x}$
9	a) $\int \mathrm{ctg}^3 x dx$	b) $\int \frac{dx}{3+5\cos x}$
10	a) $\int \frac{\sin x}{\sqrt[3]{\cos^2 x}} dx$	b) $\int \frac{dx}{1-3\cos^2 x}$
11	a) $\int \sin x \cos 3x dx$	b) $\int \frac{\sin x dx}{2-\sin x}$
12	a) $\int \frac{\sin^3 x}{\cos^2 x} dx$	b) $\int \frac{dx}{3+2\cos x}$
13	a) $\int \sin^2 x \cos^3 x dx$	b) $\int \frac{dx}{2-\cos^2 x}$
14	a) $\int \sin^3 x dx$	b) $\int \frac{\sin x dx}{2-5\sin x}$
15	a) $\int \sin 7x \sin 3x dx$	b) $\int \frac{dx}{3-\sin^2 x}$

Table 7 (the end)

1	2	3
16	a) $\int \mathrm{tg}^3 x dx$	b) $\int \frac{dx}{3-4\cos x}$
17	a) $\int \frac{\cos x}{\sqrt{\sin^3 x}} dx$	b) $\int \frac{dx}{2+3\cos^2 x}$
18	a) $\int \sin^4 x \cos^3 x dx$	b) $\int \frac{dx}{1+2\cos x}$
19	a) $\int \mathrm{tg}^2 x dx$	b) $\int \frac{dx}{2+\cos^2 x}$
20	a) $\int \sin x \cos^3 x dx$	b) $\int \frac{dx}{3+\cos x}$
21	a) $\int \cos^2 x dx$	b) $\int \frac{dx}{2-\sin^2 x}$
22	a) $\int \mathrm{tg}^4 x dx$	b) $\int \frac{dx}{2+8\cos^2 x}$
23	a) $\int \sin 2x \cos 3x dx$	b) $\int \frac{\sin x dx}{2-7\sin x}$
24	a) $\int \sin^2 x \cos^7 x dx$	b) $\int \frac{dx}{9-\cos^2 x}$
25	a) $\int \frac{\sin^5 x}{\cos^2 x} dx$	b) $\int \frac{dx}{5+2\cos x}$
26	a) $\int \sin^5 x dx$	b) $\int \frac{\sin x dx}{2+9\sin x}$
27	a) $\int \sin 8x \sin 6x dx$	b) $\int \frac{dx}{3+\sin^2 x}$
28	a) $\int \mathrm{ctg}^4 x dx$	b); $\int \frac{dx}{3-9\cos x}$
29	a) $\int \frac{\cos x}{\sqrt[7]{\sin^6 x}} dx$	b) $\int \frac{dx}{2+9\cos^2 x}$
30	a) $\int \cos^4 x dx$	b) $\int \frac{dx}{2+5\sin^2 x}$

Task 5. Find the integrals involving roots.

Table 8

1	2	3
1	a) $\int \frac{\sqrt[3]{2x-1}-3}{\sqrt[3]{2x-1}-1}dx$	b) $\int \sqrt{x^2-36}\,dx$
2	a) $\int \frac{\sqrt{4x+1}+2}{1-\sqrt{4x+1}}dx$	b) $\int x^2\sqrt{x^2+25}\,dx$
3	a) $\int \frac{\sqrt{3x-2}+1}{\sqrt[3]{3x-2}-4}dx$	b) $\int \sqrt{36-x^2}\,dx$
4	a) $\int \frac{\sqrt{5x+1}+3}{1-\sqrt[4]{5x+1}}dx$	b) $\int \sqrt{x^2+4}\,dx$
5	a) $\int \frac{\sqrt[3]{7x+1}+2}{\sqrt[3]{7x+1}-1}dx$	b) $\int \frac{\sqrt{x^2-9}}{x}dx$
6	a) $\int \frac{\sqrt{4-x}+7}{3-\sqrt{4-x}}dx$	b) $\int \sqrt{x^2-25}\,dx$
7	a) $\int \frac{\sqrt{x-8}+3}{\sqrt[3]{x-8}-4}dx$	b) $\int x^2\sqrt{36+x^2}\,dx$
8	a) $\int \frac{\sqrt{9x-1}+2}{2-\sqrt[4]{9x-1}}dx$	b) $\int \sqrt{9-x^2}\,dx$
9	a) $\int \frac{\sqrt[3]{4x+5}-8}{\sqrt[3]{4x+5}+2}dx$	b) $\int \sqrt{x^2+16}\,dx$
10	a) $\int \frac{\sqrt{1-2x}+3}{5-\sqrt{1-2x}}dx$	b) $\int x^2\sqrt{16+x^2}\,dx$
11	a) $\int \frac{\sqrt{x+6}+5}{\sqrt[3]{x+6}-7}dx$	b) $\int \sqrt{x^2+121}\,dx$
12	a) $\int \frac{\sqrt{9-x}+4}{8-\sqrt[4]{9-x}}dx$	b) $\int \sqrt{x^2-16}\,dx$
13	a) $\int \frac{1-\sqrt[3]{2x+5}}{\sqrt[3]{2x+5}-3}dx$	b) $\int \frac{\sqrt{x^2-1}}{x}dx$
14	a) $\int \frac{\sqrt{1-2x}+1}{\sqrt{1-2x}+5}dx$	b) $\int \sqrt{4-x^2}\,dx$
15	a) $\int \frac{\sqrt{3x-7}+2}{1-\sqrt[3]{3x-7}}dx$	b) $\int x^2\sqrt{9+x^2}\,dx$

Table 8 (the end)

1	2	3
16	a) $\int \frac{\sqrt{8x-1}+1}{9-\sqrt[4]{8x-1}}dx$	b) $\int \frac{\sqrt{9-x^2}}{x^2}dx$
17	a) $\int \frac{\sqrt{1-6x}+1}{\sqrt{1-6x}+8}dx$	b) $\int \frac{\sqrt{x^2+16}}{x}dx$
18	a) $\int \frac{\sqrt{2x-7}+2}{4-\sqrt[3]{2x-7}}dx$	b) $\int \sqrt{x^2-1}\,dx$
19	a) $\int \frac{\sqrt{7x-2}+1}{3-\sqrt[4]{7x-2}}dx$	b) $\int \frac{\sqrt{x^2-4}}{x}dx$
20	a) $\int \frac{\sqrt{4-3x}}{\sqrt{4-3x}+5}dx$	b) $\int \sqrt{16-x^2}\,dx$
21	a) $\int \frac{\sqrt{8-x}+5}{\sqrt{9-x}+6}dx$	b) $\int x^2\sqrt{1+x^2}\,dx$
22	a) $\int \frac{\sqrt{x-7}+2}{1+\sqrt[3]{x-7}}dx$;	b) $\int \frac{\sqrt{1-x^2}}{x^2}dx$
23	a) $\int \frac{\sqrt{x-2}+6}{3-\sqrt[4]{x-2}}dx$	b) $\int \frac{\sqrt{x^2+4}}{x}dx$
24	a) $\int \frac{\sqrt{x+6}}{\sqrt{x+6}+10}dx$	b) $\int \sqrt{x^2-9}\,dx$
25	a) $\int \frac{\sqrt[3]{3x-1}+9}{\sqrt[3]{3x-1}-4}dx$	b) $\int \frac{\sqrt{x^2-9}}{x}dx$
26	a) $\int \frac{\sqrt{7x+1}+2}{4-\sqrt{7x+1}}dx$	b) $\int \sqrt{25-x^2}\,dx$
27	a) $\int \frac{\sqrt{x-2}+1}{\sqrt[3]{x-2}-4}dx$	b) $\int x^2\sqrt{4+x^2}\,dx$
28	a) $\int \frac{\sqrt{1-x}+3}{4+\sqrt[4]{1-x}}dx$	b) $\int \frac{\sqrt{9-x^2}}{x^2}dx$
29	a) $\int \frac{\sqrt{x}+8}{3-\sqrt[3]{x}}dx$	b) $\int \frac{\sqrt{x^2+16}}{x}dx$
30	a) $\int \frac{\sqrt{6-x}}{\sqrt[4]{6-x}+5}dx$	b) $\int \sqrt{x^2-4}\,dx$

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Content

НАВЧАЛЬНЕ ВИДАННЯ

ВИЩА МАТЕМАТИКА

Методичні рекомендації до самостійної роботи за темою "Інтегральне числення" для студентів галузі 12 "Інформаційні технології" першого (бакалаврського) рівня (англ. мовою)

Самостійне електронне текстове мережеве видання

Укладачі: **Рибалко** Антоніна Павлівна
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Викладено необхідний теоретичний матеріал з навчальної дисципліни та наведено типові приклади, які сприяють найбільш повному засвоєнню матеріалу з теми "Інтегральне числення" та застосуванню отриманих знань на практиці. Наведено детальний опис та методичні рекомендації до виконання завдань для самостійної роботи, перелік літературних джерел, теоретичні запитання та тест для самодіагностики з метою вдосконалення знань студентів за даною темою. Визначено професійні компетентності, яких набувають студенти в результаті вивчення теоретичного матеріалу та виконання практичних завдань за цією темою.

Рекомендовано для студентів галузі 12 "Інформаційні технології" першого (бакалаврського) рівня.

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