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Construction of canal surfaces based on a specified flat curvature line

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Abstract. The research relevance is predefined by the widespread use of elements and structures that have the shape of canal surfaces in engineering practice and the possibilities of reproducing the surface through kinematics. The research aims to develop new means of modeling canal surfaces referred to as a grid of curvature lines by introducing elements with special properties into the structural model that simplify the solution of differential equations and reduce the amount of computation. To achieve the research methods, the synthetic geometric method, methods of linear algebra, the theory of differential equations and differential geometry, as well as methods of computer geometric modeling and visualization of three-dimensional objects were used. Studies on modeling and studying the properties of channel surfaces are analyzed. The research on the problem of the surfaces and lines of curvature is considered in more detail and the conditions under which it is possible to simplify the solution of the differential equation are identified. It was proved that the condition of contact between the canal surface and the plane along a given plane curve is sufficient for this curve to be one of the curvature lines of the family of orthogonal to the generating circles. This allowed to reduce the solution of the differential equation to two quadratures. The expressions of the corresponding integrals and an algorithm for modeling the canal surface with the possibility of referring to a grid of curvature lines were obtained. The expressions that define the desired surface include the parametric equations of a given plane line; a function that determines the radii of the spheres of the family depending on the parameter of this line. A specific example of modeling a surface based on a defined formula was also considered, and images of this surface with visualization of the coordinate grid were presented. The research's practical values are defined by the possibility of using the developed modeling tools in the design and computer-aided design of the geometry of real products containing surfaces of a smooth transition of variable radius

Keywords: surface coordinate grid, geometric modeling, sphere, envelope, Riccati's equation

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INTRODUCTION

Since canal surfaces can be formed kinematically, they are common in engineering practice, from mechanical and hydraulic devices to architectural structures. Information and software for geometric modeling and design consist of many tools that allow the formation of objects of the required shape. These tools can form initial shapes by moving along a given trajectory, deform (mix surfaces), reconstruct shapes, form transition surfaces between given surfaces, etc. The so-called tubular surfaces are common in geometric computer modeling, which are canal surfaces with a generating circle of constant radius and has many practical applications. For example, they can represent the surface of a spring (a perfect spiral), which can be seen in real life on climbing plants, corkscrews, etc. Such surfaces can also use the form of compartmentalized surfaces of round cylinders. Within canal surfaces, it is possible to emphasize mixing surfaces or smooth connections between two given surfaces. They are obtained from the initial canal surface through an operation consisting of generating one or more auxiliary surfaces that create a differentiated (smooth) transition between them so that the final object is a combination of all previous surfaces into a single part.

Thus, the study of new means of designing canal surfaces is necessary for the further development of computer geometric modeling tools and their implementation in computer-aided design systems for complex objects.

The research aims to develop new tools for modeling canal surfaces based on curvature lines by introducing

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elements with special properties into the structural model that simplify the solution of differential equations and reduce the computation required.

The research novelty is defined by the use of new constructive means (a specified flat line of curvature) in the geometric model of the canal surface instead of the traditional ones (sphere family center line). The introduction of the curvature line into the model allowed simultaneously controlling the surface shape and simplifying the transition to the mesh from the curvature lines on the surface, due to the existence of a known partial solution of the differential equation.

LITERATURE REVIEW

Numerous studies analyzed canal surfaces, covered various aspects of their formation and properties, identified special cases or classes of such surfaces, and created or improved tools for their modeling.

Researchers [1; 2] focus on the visualization of canal surfaces by various means based on the specified parametric equations. An overview of the applications of canal surfaces in computer geometric modeling of objects is given in [3]. In [4], the design of architectural forms by surfaces with flat curvature lines, including canal surfaces, was considered.

Canal surfaces are a specific cyclic surface case, the standard formation formula of which is stated in [5].

Several studies analyzed the development of tools for modeling canal surfaces. Study [6] considers the problem of transition from representing the canal surface in a parametric form relative to the line of sphere centers to an implicit form of representing the surface by a system of equations. In [7], a special parameterization of the guide curve was proposed using a rational function. In [8], the Bezier curve was chosen as the guideline for the canal surface, and in [9], a hyperbolic curve was chosen.

The study of the properties of canal surfaces was considered in [10; 11]. In [10], the symmetry properties that canal surfaces with rational center curves and the function of changing the radius of the current parent sphere can have were studied. The study [11] considers canal surfaces obtained by conformal transformations.

The study of canal surfaces in four-dimensional space is considered in [12; 13], [14; 15] – in the Minkowski space, in [16] – in the pseudo-Halley three-dimensional space, and in [17] – in the spherical geometry of C. Li.

Among the canal surfaces, it is possible to distinguish special classes that satisfy additional conditions imposed on the geometric characteristics of the surfaces. For example, in [18], canal surfaces that retain the mean curvature under isometric transformations were considered, and [19; 20] studied canal surfaces that are also Weingarten surfaces.

In [21], the mathematical apparatus of quaternions was applied to the modeling of canal surfaces.

The problem of assigning a canal surface to the curvature lines is obtaining a family of lines orthogonal to the characteristic circles on the surface. This problem was considered in general in [22] and is reduced to solving an ordinary first-order differential equation of the Riccati type, which is not integrated into quadratures. This determines the need to study special cases of canal surfaces that allow solving the differential equation of curvature lines in quadratures or to numerically integrate this equation. For example, if all spheres of a family have the same radius (tubular surface) or the line of centers of the spheres of the family is a flat curve, the curvature lines can be obtained using a single quadrature or without integration at all. Analyzing the known properties of the Riccati differential equation and comparing them with the above studies, the construction of solutions to this equation can be associated with the problem of the curvature line of the family, which is orthogonal to the characteristic circles. Thus, if one partial solution of the Riccati equation is known (one given curvature line), the general solution is found using two quadratures [23], and, therefore, the assignment of two curvature lines requires one quadrature to describe the general solution. Finally, the three known partial solutions of this equation allow us to write the general solution without quadratures.

MATERIAL AND METHODS

In the first research stage, a mathematical model of the canal surface was considered in a generalized parametric form (the "classical" model), when the surface determinant includes a guideline (line of sphere centers) and a function of the dependence of the radius of the current sphere of the family on the guide parameter. Using the differential geometry methods, it was proved that the coordinate grid on the canal surface obtained by this model is not a grid of curvature lines. It is demonstrated that the transition to such a grid requires solving a Riccati-type differential equation. Known cases when this equation has a quadratic solution are analyzed.

In the second stage, a synthetic geometric method was used to prove the existence of a curvature flat line on the canal surface, provided that the plane touches the surface along this curve. Using the methods of differential calculus, it was verified that it is possible to form one partial solution of the differential equation based on this curve. Based on the obtained partial solution, a general solution of the Riccati-type differential equation was constructed using the methods of the theory of ordinary differential equations, which allowed us to find the lines of the family orthogonal to the generating circles of the canal surface and to refer the surface to the lines of curvature.

In the third stage, based on the obtained mathematical model, images of the surface referred to as the grid of curvature lines were obtained by computer visualization methods using the tools of the Matplotlib library of the Python programming language.

RESULTS AND DISCUSSION

A canal is the envelope of a one-parameter family of spheres. The characteristic curves of a family of spheres are circles that create one of two families of curvature lines on this surface.

The canal surface equation in a vector form will be presented

$$\boldsymbol{m} = \boldsymbol{\rho} + a\,\boldsymbol{\tau} + r\ \cos u\ \boldsymbol{\nu} + r\ \sin u\ \boldsymbol{\beta}, \tag{1}$$

in which ρ=$\rho(t)$ – radius is a vector of a point on the line of centers of the family of spheres; τ, ν, β – unit vectors (orthos) of the Fresnel system of this line; t is a parameter that determines the position of the point on the guideline; u is a parameter of the position of the point on the generating circle; a=$a(t)$ is a function that determines the distance from the point of contact on the line of centers of the spheres to the center of the generating circle along the tangent, and has the form:

$$a = -\frac{RR'}{|\boldsymbol{\rho}'|}, \quad (2)$$

in which R=R(t) is the function of the dependence of the radius of the family sphere on the parameter of the guideline; $|\boldsymbol{\rho}'| = \left|\frac{d\rho}{dt}\right|$ – the modulus of the tangent vector to the guideline; r=r(t) is the function that establishes the dependence of the radius of the generating circle on the parameter of the guideline:

$$\boldsymbol{r} = \frac{\boldsymbol{R}|\boldsymbol{\rho}'|}{\sqrt{|\boldsymbol{\rho}'|^2-\boldsymbol{R}'^2}}. \quad (3)$$

Find the partial derivatives of function (1):

$$\frac{\partial \boldsymbol{m}}{\partial t} = (|\boldsymbol{\rho}'| + a' - r|\boldsymbol{\rho}'|k\ \cos u)\ \boldsymbol{\tau} + (a|\boldsymbol{\rho}'|k + r'\cos u - r|\boldsymbol{\rho}'|\upsilon\ \sin u)\ \boldsymbol{\nu} + (r|\boldsymbol{\rho}'|\upsilon\ \cos u + r'\sin u)\ \boldsymbol{\beta},$$
$$\frac{\partial \boldsymbol{m}}{\partial u} = -r\ \sin u\ \ \boldsymbol{\nu} + r\ \cos u\ \ \boldsymbol{\beta}, \frac{\partial^2 \boldsymbol{m}}{\partial u^2} = -r\ \cos u\ \ \boldsymbol{\nu} - r\sin u\ \ \boldsymbol{\beta}, \quad (4)$$
$$\frac{\partial^2 \boldsymbol{m}}{\partial u\ \partial t} = r|\boldsymbol{\rho}'|k\sin u\ \ \boldsymbol{\tau} - (r'\sin u + r|\boldsymbol{\rho}'|\upsilon\ \cos u)\ \boldsymbol{\nu} + (r'\cos u - r|\boldsymbol{\rho}'|\upsilon\ \sin u)\boldsymbol{\beta},$$

in which a', r' are the derivatives of the corresponding functions, k=k(t) is the curvature of the guideline; υ is the twist of the guideline.

First quadratic form coefficients

$$E = \left(\frac{\partial \boldsymbol{m}}{\partial u}\right)^2 = r^2, F = \left(\frac{\partial \boldsymbol{m}}{\partial u}\cdot\frac{\partial \boldsymbol{m}}{\partial t}\right) = r^2|\boldsymbol{\rho}'|(\upsilon - \frac{a}{r}k\sin u). \quad (5)$$

The numerators of the coefficients M and L of the second quadratic form

$$M' = \frac{\partial^2 \boldsymbol{m}}{\partial u \partial t}\left[\frac{\partial \boldsymbol{m}}{\partial u}\frac{\partial \boldsymbol{m}}{\partial t}\right] = -r^2|\boldsymbol{\rho}'|(r'k\ \ \sin u - a\ k^2|\boldsymbol{\rho}'|\sin u\ \ \cos u - \upsilon(|\boldsymbol{\rho}'| + a' - r\ k|\boldsymbol{\rho}'|\cos u)),$$
$$L' = \frac{\partial^2 \boldsymbol{m}}{\partial u^2}\left[\frac{\partial \boldsymbol{m}}{\partial u}\frac{\partial \boldsymbol{m}}{\partial t}\right] = -r^2(|\boldsymbol{p}'| + a' - r\ k|\boldsymbol{\rho}'|\cos u). \quad (6)$$

The conditions F=0 and M=0 are not fulfilled. In other words, in general, the coordinate grid of the surface (1) does not coincide with the lines of curvature. More precisely, the lines t=const (generating circles) make up one set of curvature lines, as for the lines u=const, they are not curvature lines in general, since the conditions of orthogonality and conjugacy of the coordinate grid are not met.

To find the curvature lines of the family orthogonal to the generating circles, the differential equation of orthogonal trajectories to the family t=const will be used, which has the form [24]:

$$Fdu + Edt = 0. \quad (7)$$

Substituting the expressions E and F into equation (7), after reductions, the following equations were obtained

$$\frac{du}{dt} = \frac{a}{r}k|\boldsymbol{\rho}'|\sin u - |\boldsymbol{\rho}'|\upsilon. \quad (8)$$

Replacing the variable u in the last equation with

$$v = tg\frac{u}{2} \quad (9)$$

resulted in the generalized Riccati equation [23].

$$v' = -\frac{|\rho'|\upsilon}{2}v^2 + \frac{a}{r}k|\boldsymbol{\rho}'|v - \frac{|\rho'|\upsilon}{2}. \quad (10)$$

Substituting the values of a and r from the right-hand side of equations (2) and (3) into (8) and (10), the following is true:

$$\frac{du}{dt} = -\frac{R'}{\sqrt{|\boldsymbol{\rho}'|^2-R'^2}}k|\boldsymbol{\rho}'|\sin u - |\boldsymbol{\rho}'|\upsilon, \quad (11)$$

$$v' = -\frac{|\rho'|\upsilon}{2}(v^2+1) - \frac{R'k|\rho'|}{\sqrt{\rho'^2-R'^2}}v. \quad (12)$$

It is assumed, that function

$$v = v(t, \alpha), \quad (13)$$

is an equation (12) solution, where α is the integral constant. Then, using equality (9), the following expression is true

$$u = u(t, \alpha), \quad (14)$$

in which, when substituted into equation (1), will transform it into the equation of the canal surface referred to as the coordinate grid of curvature lines t=const, α=const.

The four-solution property of the Riccati equation [23] allows to state that the four curvature lines of the family orthogonal to the generating circles intersect these circles at four points with a constant anharmonic ratio. Note that if the ratio $\frac{a}{r}$ in (8) is replaced by

$$\frac{a}{r} = tg\gamma, \quad (15)$$

in which γ is the angle of inclination of the vector directed from any point of the guideline to the points of the corresponding generating circle, then equation (8) can be rewritten as

$$tg\gamma = \frac{k|\rho'|\sin u}{u'+|\rho'|\upsilon}. \quad (16)$$

Geometrically, this equation is a condition for the normal to the canal surface along the second family of curvature lines to form scanning surfaces, with line (1) serving as the guide curve.

Since the Riccati equation in the form (11) or (12) does not have a general solution in quadratic form, it was advisable to consider finding the curvature lines for specific cases of representing canal surfaces.

Thus, for tubular surfaces (R=const) with the spatial line of the centers of the family of spheres, equation (11) was obtained as

$$\frac{du}{dt} = -|\rho'|\upsilon. \quad (17)$$

As such, function (14) becomes

$$u = -\int|\rho'|\ \ \upsilon\ dt + \alpha. \quad (18)$$

If, in the case of tubular surfaces, the line of centers is flat (υ=0), then according to (18) $\frac{du}{dt}=0$, the coefficients F and M of the first and second quadratic forms will be zero. Consequently, in this case, both the t=const and u=const lines of the surface coincide with the curvature lines.

If the dependence for the radius of the sphere is variable, and the line of centers remains a flat curve (R≠const, υ=0), then the solution of equation (11) reduces to quadrature:

$$u = 2\ arctg\, \alpha\ \ e^{-\int \frac{R'}{\sqrt{|\rho'|^2 - R'}}|\rho'|k dt}. \tag{19}$$

Note that the values u=0 and u=π, which correspond to the cross-section of the surface by the plane of the line of centers, give two lines of curvature since the normal to the surface along these lines belong to the same plane.

In the case when R≠const, υ=0, and k=0 is a straight lines, equation (1) gives a surface of rotation in which the lines t=const are meridians and the lines u=const are parallels.

As for the case when R≠const, υ≠0, the radius of the spheres is variable and the line of centers is a spatial curve, the solution of equation (12) is possible only for some special cases [23]. Therefore, if (12) has no solution in quadratures or the quadratures obtained from it have no analytical representation, numerical integration methods can be used to obtain a coordinate grid from the curvature lines, which leads to a significant increase in the amount of computation.

The following geometric model of the canal surface formation is proposed. A certain line C lying on the Oxy plane of the rectangular spatial coordinate system was considered. This plane is called the reference plane. The one-parameter set of spheres was determined from the condition that the spheres of the family and the reference plane touch along the given line.

It has been proved that this condition is sufficient for a given plane line to be a line of curvature on the contour surface of a family of spheres if one exists. According to the transitivity theorem [24], if a surface (S) has a contact with two surfaces (Σ1) and (Σ2) of order at least n at the same point, then (Σ1) has (Σ2) a contact of the corresponding order at the same point. Taking the surface (S) to be the sphere of the family, and the surfaces (Σ1) and (Σ2) to be the reference plane and the canal envelope, we conclude that, if it exists, this surface will touch the plane along the points of the given curve. Considering touching as a limiting case of intersection of surfaces, and given that any line lying on a plane is its line of curvature, according to Joachimsthal's theorem [24], we conclude that a given plane curve is also a line of curvature for the canal surface.

In this case, the condition for the contact of an arbitrary sphere of the family and the Oxy plane was determined from the equation for the line of centers of the spheres in the form:

$$\boldsymbol{\rho} = f\ \boldsymbol{i}\ + \varphi\ \boldsymbol{j} + R\ \boldsymbol{k}, \tag{20}$$

where: f=$f(t)$, φ=$\varphi(t)$ are functions of the parameter t that define a given line on the Oxy plane, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are the orthoi of the rectangular Cartesian coordinate system.

Then, the vector equation of the canal surface (1) is as follows:

$$\boldsymbol{m} = \boldsymbol{\rho} + \frac{RR'}{\sqrt{\boldsymbol{\rho}'^2}}\boldsymbol{\tau} + R\sqrt{\frac{\boldsymbol{\rho}'^2 - R'^2}{\boldsymbol{\rho}'^2}}\ (cos\, u\ \ \boldsymbol{\nu} + sin\, u\ \ \boldsymbol{\beta}). \tag{21}$$

By substituting the known expressions for calculating the vectors $\boldsymbol{r}, \boldsymbol{r'}, \boldsymbol{\tau}, \boldsymbol{n}, \boldsymbol{\beta}$, and performing the transformation, the following parametric equations are obtained:

$$\begin{gathered} x = \frac{1}{\boldsymbol{\rho}'^2|\boldsymbol{\rho}'\times\boldsymbol{\rho}''|}(f\ \boldsymbol{\rho}'^2|\boldsymbol{\rho}'\times\boldsymbol{\rho}''| + R[\sqrt{f'^2+\varphi'^2}(cos\, u\, (f''(\varphi'^2+R'^2) - f'(\varphi'\varphi''-R'R'')) + sin\, u\, |\boldsymbol{\rho}'|(\varphi'R''-R'\varphi'')) - R'f'|\boldsymbol{\rho}'\times\boldsymbol{\rho}''|]), \\ y = \frac{1}{\boldsymbol{\rho}'^2|\boldsymbol{\rho}'\times\boldsymbol{\rho}''|}(\varphi\ \boldsymbol{\rho}'^2|\boldsymbol{\rho}'\times\boldsymbol{\rho}''| + R[\sqrt{f'^2+\varphi'^2}(cos\, u\, (\varphi''(f'^2+R'^2) - \varphi''(f'f''-R'R'')) - sin\, u\, |\boldsymbol{\rho}'|(f'R''-R'f'')) - R'\varphi'|\boldsymbol{\rho}'\times\boldsymbol{\rho}''|]), \\ z = \frac{R}{\boldsymbol{\rho}'^2|\boldsymbol{\rho}'\times\boldsymbol{\rho}''|}((f'^2+\varphi')|\boldsymbol{\rho}'\times\boldsymbol{\rho}''| + \sqrt{f'^2+\varphi'^2}\ [cos\, u\, (R''(f'^2+\varphi'^2) - R''(f'f''-\varphi'\varphi'')) + sin\, u\, |\boldsymbol{\rho}'|(f'\varphi''-\varphi'f'')]), \end{gathered} \tag{22}$$

in which: $|\boldsymbol{\rho}'| = \sqrt{\boldsymbol{\rho}'^2} = \sqrt{f'^2+\varphi'^2+R'^2}$, $f', f'', \varphi', \varphi'', R', R''$ are the first and second derivatives of the corresponding functions, $\boldsymbol{\rho}'' = \frac{d^2\boldsymbol{\rho}}{dt^2}$ and $|\boldsymbol{\rho}'\times\boldsymbol{\rho}'| = \sqrt{(\varphi'R''-\varphi''R')^2 + (f'R''-R'f'')^2 + (f'\varphi''-\varphi'f'')^2}$.

As noted earlier, the surface constructed according to equation (22) will not be classified as a curvature line, since only one of its two families of coordinate lines is a curvature line. These are the lines of t=const, which are the generating circles. To move to the system of curvature lines, it is necessary to move in equations (22) from the parameter u to a new parameter obtained from differential equation (12) concerning (9).

Using the known correlations for curvature and torsion, this equation can be remade as:

$$\upsilon' = -\frac{|\boldsymbol{\rho}'|(\boldsymbol{\rho}'\boldsymbol{\rho}''\boldsymbol{\rho}''')}{2(\boldsymbol{\rho}'\times\boldsymbol{\rho}'')^2}(\upsilon^2+1) - \frac{R'\ |\boldsymbol{\rho}'\times\boldsymbol{\rho}''|}{\boldsymbol{\rho}'^2\sqrt{\boldsymbol{\rho}'^2-R'^2}}\upsilon, \tag{23}$$

where: $(\boldsymbol{\rho}'\boldsymbol{\rho}''\boldsymbol{\rho}''')$ – is the mixed product of the first, second, and third derivatives of the vector (20), $\boldsymbol{\rho}'\times\boldsymbol{\rho}''$ – is the vector product of the first and second derivatives of the vector ρ.

Since the flat line C is, as proved above, the line of curvature of the canal surface, the corresponding expression for the parameter v must satisfy the differential equation (12) and, therefore, be its partial solution. Note that the reference plane in the coordinate system is characterized by the equation – z=0, then by equating the third of equations (22) to zero and making the appropriate replacement of the parameter, we obtain the equation quadratic concerning v:

$$[|\boldsymbol{\rho}'\times\boldsymbol{\rho}''|\sqrt{f'^2+\varphi'^2} - R''(f'^2+\varphi') + R'(f'f''+\varphi'\varphi'')]\upsilon^2 + 2|\boldsymbol{\rho}'|(f'\varphi''-\varphi'f'')\upsilon + |\boldsymbol{\rho}'\times\boldsymbol{\rho}''|\sqrt{f'^2+\varphi'^2} + R''(f'^2+\varphi') - R'(f'f''+\varphi'\varphi'') = 0. \tag{24}$$

This equation has a discriminant equal to zero and yields the desired partial solution, v_1=$v_1(t)$:

$$\upsilon_1 = |\boldsymbol{\rho}'|(\varphi'f''-f'\varphi'')\ [|\boldsymbol{\rho}'\times\boldsymbol{\rho}''|\sqrt{f'^2+\varphi'^2} - R''(f'^2+\varphi'^2) + R'(f'\ f''+\varphi'\varphi'')]^{-1}. \tag{25}$$

Directly substituting the right-hand side of (25) for v, as well as the derivative of this expression for the differential equation (12), reduces the latter to zero, which confirms the correctness of the result.

It is known from the theory of differential equations [23] that given one partial solution of the Riccati differential equation, its general solution can be obtained using two quadratures.

For equation (12) the general solution is

$$v = v_1 + \frac{1}{\phi(t)(w + \psi(t))}, \tag{26}$$

where: w is the new variable (parameter) for which the integration constant is taken; $\phi(t)$ is the function obtained from the expression:

$$\phi(t) = e^{\int\left(\frac{|\rho'|(\rho'\rho'\rho''')}{(|\rho'\times\rho''|)^2} v_1 + \frac{R'|\rho'\times\rho''|}{\rho'^2\sqrt{\rho'^2 - R'^2}}\right)dt}; \tag{27}$$

ψ(t) is a following function:

$$\psi(t) = \int\left[\frac{|\rho'|(\rho'\rho''\rho''')}{2(\rho'\times\rho'')^2}\, e^{-\int\left(\frac{|\rho'|(\rho'\rho''\rho''')}{(\rho'\times\rho'')^2} v_1 + \frac{R'|\rho'\times\rho''|}{\rho'^2\sqrt{\rho'^2 - R'^2}}\right)dt}\right] dt. \tag{28}$$

To move from the original grid on the surface to the coordinate grid of curvature lines, based on the results obtained, it is necessary to replace the parameter u in equation (22) based on the following equation:

$$u = 2\,arctan(v), \tag{29}$$

in which $v = v(t, w)$ has the right side expression (26).

An example of designing a canal surface using the obtained expressions is shown. As a given line of curvature, a circle was chosen, written by the parametric equations on the plane *Oxy*: $f = r\cos t$, $\varphi = r\sin t$ and the dependence function for the radius of the spheres of the family, which is as follows: $R = b\cos 2t + c$.

Figures 1 and 2 show the representations of the canal surface with a given circle of curvature (shown by the bold line in Fig. 2) obtained for two variants of the parametric surface equations: a) based on the equations (22): b) considering the solution of the differential equation (23) by expressions (26)-(28) and substitution (29).

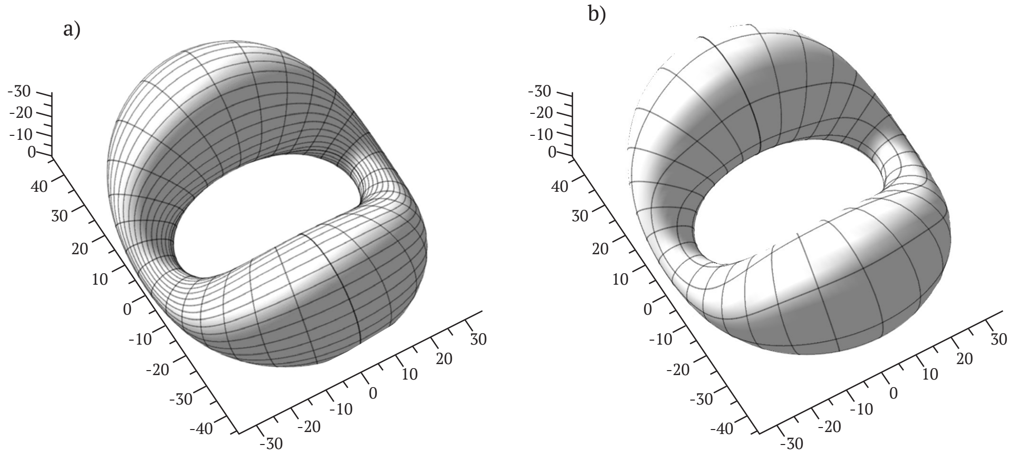

Figure 1. Canal surface (general view) at r= 3.0, b=5.5, c=10.5
a) initial parameter – (22); b) variable parameter u in relation to expressions (26), (29)

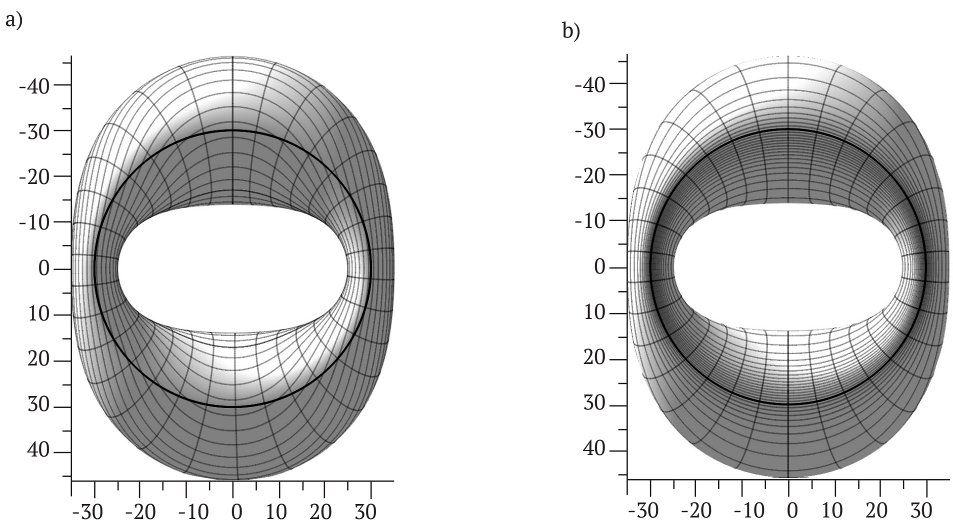

Figure 2. Distribution of coordinate grid lines on the surface near a specified plane curvature line
a) initial parameter – (22); b) variable parameter u in relation to expressions (26), (29)

Note that when constructing three-dimensional (3D) surface models, the same number of lines forming the coordinate grid was set for both parameterization options. Figure 1 is an arbitrary rectangular projection for both variants of surface visualization with the same orientation. Figure 2 shows a rectangular projection of the surface on a horizontal plane (top view).

Analyzing the results of 3D surface model visualization models built on the basis of the initial parameterization and based on the solution of the differential equation, it is possible to clearly understand the difference in the location of the coordinate grid lines on the surface. The coordinate lines of t=const (these are the generating circles) remain unchanged, while the direction of the second family of lines changes from non-orthogonal to orthogonal to the generating circles. Figure 2 well explains this difference by the fact that the given curvature guide circle intersects the lines of the u=const family on the surface (22) (Fig. 2a), and on the surface constructed with the expressions (26) and (29) (Fig. 2b), the given plane guide is part of the w=const family of lines without intersecting with other lines of the family.

The abovementioned example provides a fairly simple overview of the functions that define the plane line of curvature and the radius of the current family sphere. In more complicated cases, the expressions of the integrals in formulas (27) and (28) may not have a solution in elementary functions, so numerical integration methods will be required.

While comparing the conducted research with other studies, it is possible to state that study [22] is more theoretical in nature. Based on the study of the differential characteristics of the curvature lines of canal surfaces, the presence of no more than two isolated periodic curvature lines for a family of orthogonal generating circles was established, and the conditions for the presence of curves with ombilical points on the surface were revealed in general. In [25], the conditions and special cases of modeling canal surfaces were considered when the parameter of the guideline of the centers of the spheres is also a parameter of the curvature lines of the family orthogonal to the generating circles. Unfortunately, there are no examples of surface modeling in this paper. In [26], only certain cases of the canal surface formed by the circles of curvature of a conical helical line, as well as the case of the generating circles located in the straight plane of the guide curve, which allows integration of the differential equation of curvature lines, were considered. The main difference between the present work and these works is that the constructive model was initially proposed, the existence of which was proved by a synthetic method based on statements and theorems. The basic one is Joachimsthal's theorem. Other studies used an analytical approach when the differential equation of curvature lines was first derived, and then, by analyzing the components of this equation, possible cases were separated that allow for simplified calculations.

The main motivation for this research was to create design tools that would be more convenient for further application in practice. This tool is the inclusion of a given plane curvature line in the model. Since the constructed surface touches the plane along this line, it has the properties of a line with zero total curvature or a line of parabolic surface points. That is, it has special differential geometric properties. In contrast to the design scheme based on the representation of the line of the centers of spheres as a guide (works [6-7]), this flat line of curvature makes it much easier to coordinate the smooth transition of surfaces in 3D models of composite shapes of parts and structures.

CONCLUSIONS

The research analyzes the issue of designing canal surfaces under specified conditions, which are represented by a given flat line lying on the surface and simultaneously representing one of the curvature lines of the family of orthogonal to the generating circles. Based on the consideration of the surface determinant in the form of a line of centers of a one-parameter set of spheres and the function of the dependence of the radius of the current sphere on the parameter of the guideline, a general mathematical model of the assignment of canal surfaces to curvature lines was obtained. This model can be represented in the form of the Riccati differential equation concerning the parameter of the guideline. Important cases of simplification of the mathematical model and their geometric meaning are investigated. It was proved by a synthetic method that the condition of contact between the plane of a certain plane line and the canal surface is sufficient for this line to be a line of curvature on the surface. Based on this result, the model with a guide in the form of a line of sphere centers was modified to a model with a given plane curvature line. Based on the modified model, a partial solution of the Riccati equation was found. This allowed to reduce the problem of mapping the canal surface to a grid of curvature lines to the calculation of two quadratures instead of the need to solve the differential equation numerically. Based on the proposed mathematical model, computer simulations were carried out for a specific example of a given plane line of curvature. The results obtained are compared with the results of the most recent studies.

Summarizing the research and developing a synthetic approach to the design of canal surfaces related to curvature lines, it is possible to identify the following promising areas for further research:

- development of the proposed design scheme based on two specified plane lines of curvature, the presentation of which, under certain conditions, will further simplify the obtaining of a coordinate grid from the lines of curvature;
- expansion of the synthetic approach to the case of designing canal surfaces along a specified spherical curvature line;
- a combination of plane and spherical curvature lines in the model;
- design of the canal surface based on a specified curvature line, which is simultaneously the curvature line of the deployed surface and belongs to the family orthogonal to its generating straight lines.

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Конструювання каналових поверхонь за заданою плоскою лінією кривини

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Анотація. Актуальність дослідження обумовлюється поширеністю застосування елементів деталей та конструкцій, що мають форму каналових поверхонь, в інженерній практиці та можливостями відтворення поверхні кінематичним способом. Мета дослідження – розроблення нових засобів моделювання каналових поверхонь віднесених до сітки з ліній кривини за рахунок впровадження в конструктивну модель елементів зі спеціальними властивостями, які дозволяють спростити розв’язок диференціальних рівнянь та скоротити об’єм обчислень. Для вирішенні наукових завдань були використані синтетичний геометричний метод, методи лінійної алгебри, теорії диференціальних рівнянь та диференціальної геометрії, а також методи комп’ютерного геометричного моделювання та візуалізації тривимірних об’єктів. Проаналізовано дослідження з моделювання та вивчення властивостей каналових поверхонь. Детальніше розглянуто дослідження присвячені проблемі віднесення цих поверхонь до ліній кривини та виявлено умови, за якими можливо спростити розв’язок диференціального рівняння. Було доведено достатність умови дотику каналової поверхні та площини вдовж заданої плоскої кривої для того, щоб ця крива була однією з ліній кривини сімейства ортогонального до твірних кіл. Ця обставина дозволила звести розв’язок диференціального рівняння до двох квадратур. Були отримані вирази відповідних інтегралів та алгоритм моделювання каналової поверхні з можливостю віднесення до сітки з ліній кривини. До виразів, що визначають шукану поверхню, входять: параметричні рівняння заданої плоскої лінії; функція, що визначає радіуси сфер сімейства в залежності від параметра цієї лінії. Також був розглянутий конкретний приклад моделювання поверхні у відповідності до визначених формул, наведені зображення цієї поверхні з візуалізацією координатної сітки. Практичне значення роботи полягає у можливостях використання розроблених засобів моделювання при конструюванні та комп’ютерному проектуванні геометрії реальних виробів, що містять поверхні плавного переходу змінного радіусу

Ключові слова: координатна сітка на поверхні, геометричне моделювання, сфера, обвідна поверхня, рівняння Ріккаті