# Impact of Maturity Distribution on Dynamics of Bank‘s Term Deposits

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**Abstract**. It is shown that a program of attracting the banking term deposits essentially depends on term structure of new deposits. Solving the linear integral Volterra equations with difference kernel, it was received the explicit analytical solution for the program of attracting new deposits with an arbitrary Erlang distribution which has analytical community. The obtained solution enables to identify the modes of attracting new deposits and to develop an effective program of attracting the banking term deposits.

**Keywords**: retail term deposit, bank, balance, credit turnover, debit turnover, term to maturity, Volterra integral equation, Erlang distribution

JEL Classification: G21

# 1. Introduction

Retail term deposits are the important source of funding for Ukrainian banks. So, as of June 1, 2015 the share of term deposits of individuals in total liabilities of banks in Ukraine was 25.1%, and the one of total deposits of individuals was 33.3% (NBU, 2015). Therefore, competition between banks for retail deposits is always tense. The banks compete between themselves by setting attractive interest rates, developing new products and loyalty programs, etc. In order to develop effective programs for attracting retail deposits is needed to clearly understand mechanisms of deposit attraction, which allow finding new ways to improve the deposit management.

It is common for deposit management to set an aim to achieve the planned amount of deposit portfolio. To carry out this goal, it is necessary to establish a deposit attraction plan for deposit division: define the cash flows, credit turnover the bank should have. Achieving such aim is complicated due to dependence of deposit cash flows on its maturity term. As shown by Voloshyn (2014), a focus solely on change of deposit balances and neglecting maturity structure of deposit portfolio can lead to making wrong managerial decisions.

Admittedly, the bank deposit management through deposit cash flows is more complex than through deposit balances, but also has greater prospects for increasing its controllability. The complexity of this process is the fact that the amount of attracted deposits is sensitive to changes in market conditions and behavior of depositors. The actual dynamics of deposit balances, debit and credit turnovers in Fig. 1 illustrates such difficulty. The problem lies in the correct interpretation of the behavior of deposits and in an appropriate justification of managerial decisions. In practice we need to isolate different modes of deposit attraction and find boundaries of its usage.

## 2. Analysis of recent research

To solve the above problems, it is advisable to use the continuous-time deterministic dynamics models of deposit attraction which are scarcely explored. One can find a limited number of papers devoted to this subject (Freedman, 2004; Selyutin & Rudenko, 2013; Voloshyn, 2004, 2005, 2007, 2014). Such models are usable because they allow applying the well-developed methods of functional analysis, identifying a qualitative picture of deposit attraction, establishing the general laws of deposit dynamics and thus deepening the understanding of the mechanisms of deposit balances formation.

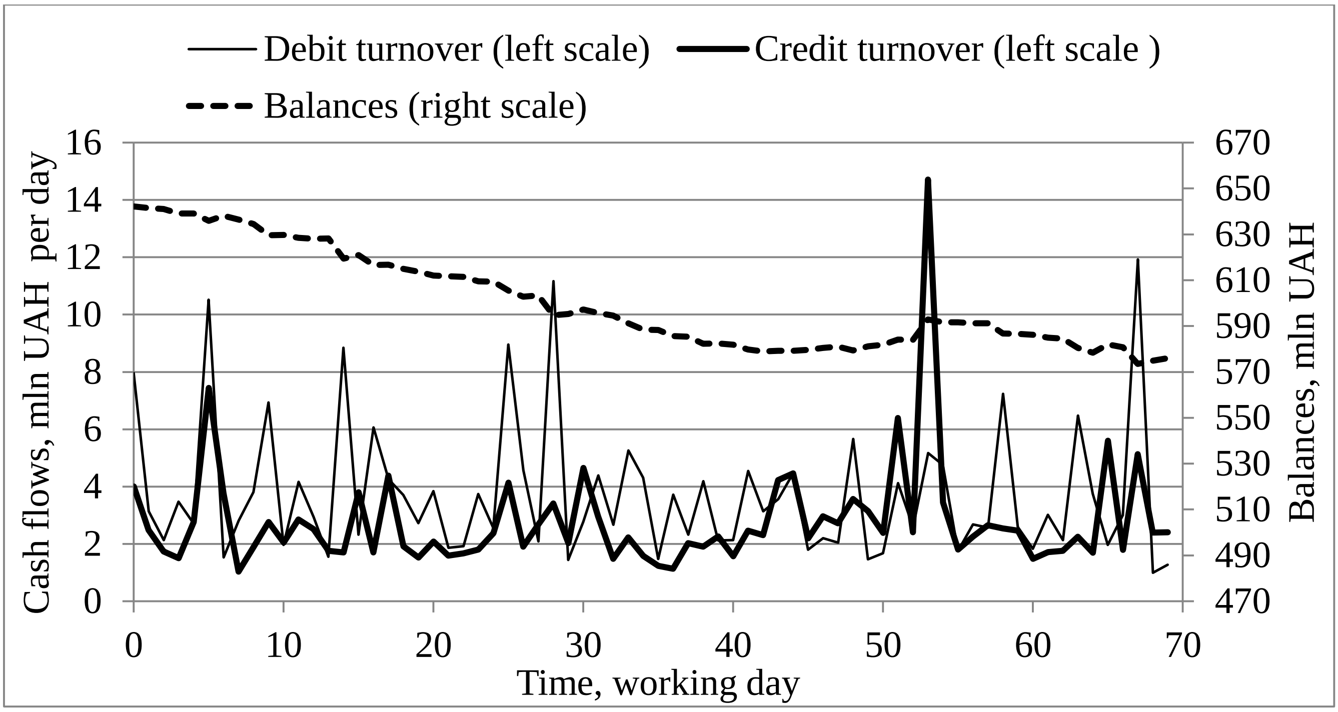


Fig. 1. An example of the actual dynamics balances, credit and debit turnovers trough deposit portfolio

In papers by Voloshyn (2004, 2005, 2007, 2014) it was investigated an influence of changes in credit turnovers and average maturity term of deposits on forming balances under exponential distribution of maturity term of deposits.

The main conclusion of the investigations (Voloshyn, 2004, 2005, 2007) is that banks mainly operate under transient regimes of liquidity. In addition, it was formulated the limit laws of conservation of cash flows during the transition from one distribution to another one that are useful for understanding long-term trends in deposit balances. An important conclusion is that the average maturity term of deposits is a necessary but insufficient liquidity early warning indicator recommended by the Basel (BIS, 2008). It was established that the dynamics of deposits is determined by the average maturities of the initial and new deposits, the initial and the new credit turnovers. These parameters exhaustively describe the dynamics of deposits.

In paper by Voloshyn (2014) it was found that the dynamics of deposits may be unobvious and lead to erroneous managerial decision making. Thus, the reduction in credit turnover with increasing the average maturity term may initially result in fall of deposit balances and then lead to their growth. This temporary decline of deposit balances may prompt the bank to increase in interest rates in order to increase in supply of deposits. Wherein, undesirable increase in interest expenses and over-fulfilment of deposit attraction plan may occur.

Conversely, increase in the credit turnover with reduction in the average maturity term of deposits can at first cause growth of deposit balances and then their fall. During the deposit growth, the bank may try to reduce interest rates that over time will reduce supply of deposits and further decline in deposit balances (Voloshyn, 2014).

However, impact of maturity structure of deposits on their dynamics has not been investigated. The article is devoted to study an impact of deposit maturity distribution on deposit attraction program.

## 1. Continuous-time model

We propose to describe density of maturity distribution of deposits through Erlang distribution (Wikipedia) (see Fig. 2):

$$f(t) = \frac{\alpha^n \cdot t^{n-1} \cdot e^{-\alpha \cdot t}}{(n-1)!}.$$

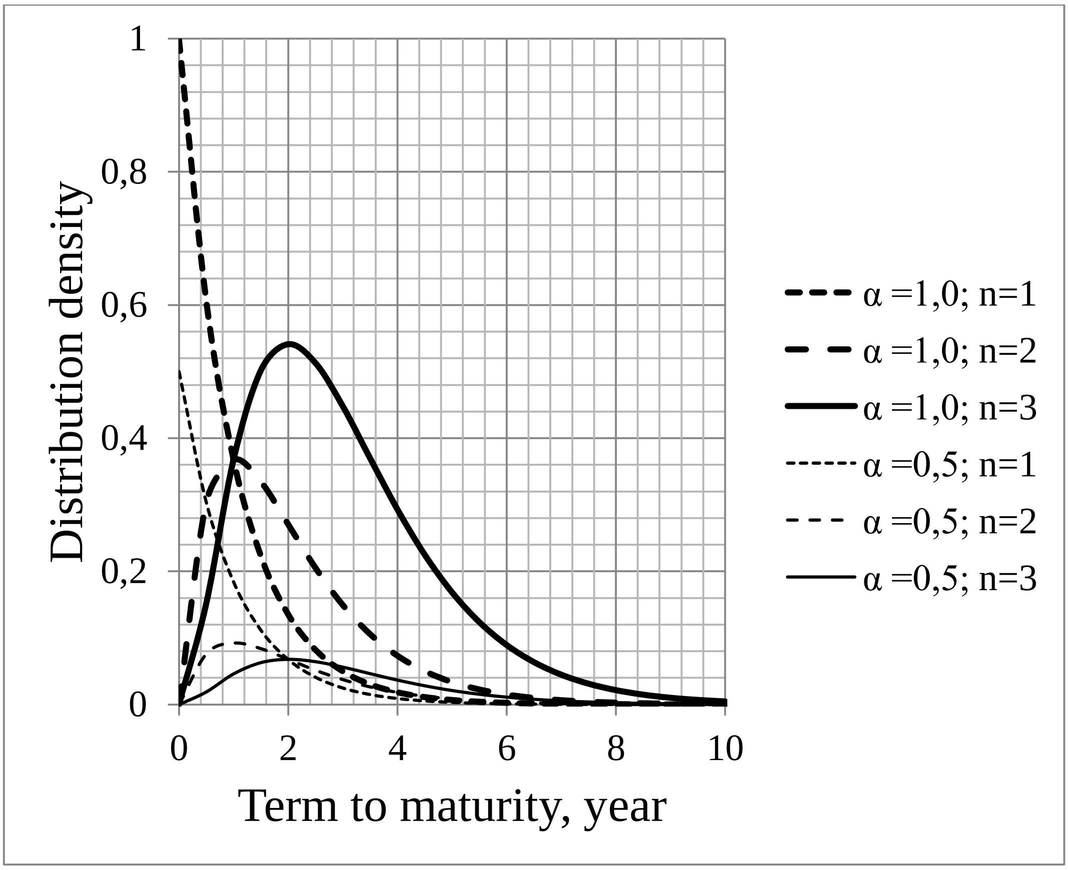


Fig. 2. Density of maturity distribution of deposits for different values of parameters $\alpha$ and $n$

Erlang distribution has only two parameters: scale $1/\alpha$ and form $n$. Medium average maturity is equal $\mu = n/\alpha$. Erlang distribution describes as convex (humped) and downward distributions that are typical for real distributions of deposit maturity. However, this distribution has convenient analytical representation that allows getting a closed analytical solution.

Let at the time $t = 0$ a bank attracts the deposits with the maturity distribution:

$$f_0(t) = \frac{\alpha_0^n \cdot t^{n-1} \cdot e^{-\alpha_0 \cdot t}}{(n-1)!}, \quad (1)$$

At the time $t > 0$ the distribution becomes:

$$f_1(t) = \frac{\alpha_1^n \cdot t^{n-1} \cdot e^{-\alpha_1 \cdot t}}{(n-1)!}, \quad (2)$$

Thus, only the scale of distribution is changed from $1/\alpha_0$ to $1/\alpha_1$, and the form of distribution remains the same $n$ = const.

As shown in paper by Voloshyn (2007) for the problem of maintaining a constant volume of deposit portfolio:

$$B = \text{const}, \tag{3}$$

dynamics of credit turnover Ct(t) describes by Volterra integral equation of the second kind:

$$Ct(t) = \varphi(t) + \int_0^t Ct(u) \cdot f_1(t-u) \cdot du\,, \tag{4}$$

where the initial maturity profile:

$$\varphi(t) = Ct_0 \cdot e^{-\alpha_0 \cdot t} \cdot \sum_{i=0}^{n-1} \frac{(\alpha_0 \cdot t)^i}{i!}\,, \tag{5}$$

and $Ct_0$ is the initial credit turnovers.

Note that the balances on deposit accounts amount is equal to:

$$B = \int_0^\infty \varphi(u) \cdot du\,, \tag{6}$$

Thus, the task is to find a program of new deposits attraction with maturity distribution (2) that helps to stabilize the deposit balances at the constant level, i.e. $B$ = const (see the condition (3)).

Integral equation (4) has a solution (Manzhirov, 2003):

$$Ct(t) = \varphi(t) + \int_0^t R(t-u) \cdot \varphi(u) \cdot du\,, \tag{7}$$

where for $k = 0,\ldots, n-1$:

$$R(t) = \frac{1}{n} \cdot e^{-\alpha_1 \cdot t} \cdot \sum_{k=0}^{n-1} e^{-\sigma_k \cdot t} \cdot \left[\sigma_k \cdot \cos(\beta_k \cdot t) - \beta_k \cdot \sin(\beta_k \cdot t)\right], \tag{8}$$

$$\sigma_k = \alpha_1 \cdot \cos\left(\frac{2 \cdot \pi \cdot k}{n}\right), \tag{9}$$

$$\beta_k = \alpha_1 \cdot \sin\left(\frac{2 \cdot \pi \cdot k}{n}\right), \tag{10}$$

Substituting expressions (9) and (10) into the formula (8) for $R(t)$, after algebraic manipulations we obtain:

$$R(t) = \frac{\alpha_1}{n} \cdot \sum_{k=0}^{n-1} e^{(\sigma_k - \alpha_1)\cdot t} \cdot \cos\left(\beta_k \cdot t + \frac{2\cdot \pi \cdot k}{n}\right), \tag{11}$$

Then the next integral becomes:

$$\int_0^t R(t-u)\cdot \varphi(u)\cdot du =$$
$$\frac{Ct_0 \cdot \alpha_1}{n} \cdot \sum_{k=0}^{n-1}\sum_{i=0}^{n-1} \frac{\alpha_0^{\,i}}{i!} \cdot e^{(\sigma_k - \alpha_1)\cdot t} \cdot \left[ I_1 \cdot \cos\left(\beta_k \cdot t + \frac{2\pi k}{n}\right) + I_2 \cdot \sin\left(\beta_k \cdot t + \frac{2\pi k}{n}\right)\right], \tag{12}$$

where

$$I_1 = \int_0^t u^i \cdot e^{(\alpha_1 - \sigma_k - \alpha_0)\cdot u} \cdot \cos(\beta_k \cdot u) \cdot du, \tag{13}$$

$$I_2 = \int_0^t u^i \cdot e^{(\alpha_1 - \sigma_k - \alpha_0)\cdot u} \cdot \sin(\beta_k \cdot u) \cdot du. \tag{14}$$

The integrals $I_1$ and $I_2$ have the following solutions (Prudnikov, Brychkov and Marichev, 2003):

$$I_1 = e^{(\alpha_1 - \sigma_k - \alpha_0)\cdot t} \cdot$$
$$\sum_{j=1}^{i+1} \frac{(-1)^{j+1} \cdot i! \cdot t^{i-j+1} \cdot \cos(\beta_k \cdot t + \varphi \cdot j)}{(i-j+1)! \cdot \left[(\alpha_1 - \sigma_k - \alpha_0)^2 + \beta_k^2\right]^{\frac{j}{2}}} - \frac{(-1)^i \cdot i! \cdot \cos(\varphi \cdot (i+1))}{\left[(\alpha_1 - \sigma_k - \alpha_0)^2 + \beta_k^2\right] \cdot \frac{i+1}{2}}, \tag{15}$$

$$I_2 = e^{(\alpha_1 - \sigma_k - \alpha_0)\cdot t} \cdot$$
$$\sum_{j=1}^{i+1} \frac{(-1)^{j+1} \cdot i! \cdot t^{i-j+1} \cdot \sin(\beta_k \cdot t + \varphi \cdot j)}{(i-j+1)! \cdot \left[(\alpha_1 - \sigma_k - \alpha_0)^2 + \beta_k^2\right]^{\frac{j}{2}}} - \frac{(-1)^i \cdot i! \cdot \sin(\varphi \cdot (i+1))}{\left[(\alpha_1 - \sigma_k - \alpha_0)^2 + \beta_k^2\right] \cdot \frac{i+1}{2}}, \tag{16}$$

where $\varphi$ is determined from the following equations:

$$\sin(\varphi) = \frac{-\beta_k}{\sqrt{(\alpha_1 - \sigma_k - \alpha_0)^2 + \beta_k^2}}, \tag{17}$$

$$\cos(\varphi) = \frac{\alpha_1 - \sigma_k - \alpha_0}{\sqrt{(\alpha_1 - \sigma_k - \alpha_0)^2 + \beta_k^2}}. \tag{18}$$

Substituting (15) and (16) into (12), and the resulting expression into (7), we obtain the desired solution for arbitrary $n$:

$$Ct(t) = Ct_0 \cdot e^{-\alpha_0 \cdot t} \cdot \sum_{i=0}^{n-1} \frac{(\alpha_0 \cdot t)^i}{i!} + \frac{Ct_0 \cdot \alpha_1}{n} \cdot \sum_{k=0}^{n-1}\sum_{i=0}^{n-1} \alpha_0{}^i \cdot \left\{ \begin{array}{l} e^{-\alpha_0 \cdot t} \cdot \sum_{j=1}^{n-1} \frac{(-1)^{j+1} \cdot t^{i-j+1} \cdot \cos\left(\frac{2\pi k}{n} - \varphi \cdot j\right)}{\left[(\alpha_1 - \sigma_k - \alpha_0)^2 + \beta_k^2\right]^{\frac{j}{2}}} - \\ - \frac{e^{(\sigma_k - \alpha_1)\cdot t} \cdot (-1)^i \cdot \cos\left(\beta_k \cdot t + \frac{2\pi k}{n} - \varphi \cdot (i+1)\right)}{\left[(\alpha_1 - \sigma_k - \alpha_0)^2 + \beta_k^2\right]^{\frac{i+1}{2}}} \end{array} \right\}. \quad (19)$$

For simplicity, but without loss of generality, consider the case when $\alpha_0 = 1$ and denoted $\alpha_1 = \alpha$. Then for the case $n = 1$ the solution (19) takes the form, which coincides with the solution obtained by Voloshyn (2007):

$$Ct(t) = Ct_0 \cdot (\alpha + (1 - \alpha) \cdot e^{-t}). \quad (20)$$

For the case $n = 2$:

$$Ct(t) = Ct_0 \cdot (A_0 + A_1 \cdot \mathrm{e}^{-2\cdot\alpha\cdot t} + A_2 \cdot \mathrm{e}^{-t} + (A_3 - A_2) \cdot t \cdot e^{-t}), \quad (21)$$

where $A_0 = \alpha$, $A_1 = (1 + \alpha)/4\alpha$, $A_2 = (-4\alpha^2 + 3\alpha - 1)/4\alpha$, $A_3 = (-3\alpha^2 + 3\alpha - 1)/2\alpha$.

For the case $n = 3$:

$$Ct(t) = Ct_0 \cdot \left\{ \begin{array}{l} A_0 + \frac{A_1 \cdot \sqrt{\frac{9\cdot\alpha^2}{4} + \left(\frac{\sqrt{3}\cdot\alpha}{2} - \frac{A_2}{A_1}\right)^2}}{\frac{3\alpha}{2}} \cdot e^{\frac{-3\cdot\alpha\cdot t}{2}} \cdot \sin\left(\frac{\sqrt{3}\cdot\alpha\cdot t}{2} - \theta\right) + \\ + e^{-t} \cdot \left[A_3 + (A_4 - 2A_3)\cdot t + (A_5 - A_4 + A_3)\cdot \frac{t^2}{2}\right] \end{array} \right\}, \quad (22)$$

where

$$\theta = arctg\left(\frac{\frac{3}{2}\alpha}{\frac{\sqrt{3}}{2}\alpha - \frac{A_2}{A_1}}\right), \quad (23)$$

$A_0 = \alpha$, $A_1 = 1 - \alpha - A_3$, $A_2 = 6\alpha^2 - 6\alpha^3 - 3\alpha^2 A_5$.The unknown coefficients $A_3$, $A_4$ and $A_5$ are determined from the matrix equation:

$$X = B^{-1} \cdot C, \qquad (24)$$

where matrixes $X$, $B$ and $C$ have the forms:

$$X = \begin{pmatrix} A_3 \\ A_4 \\ A_5 \end{pmatrix}, B = \begin{pmatrix} -1 & 3\alpha^2 & 3\alpha - 9\alpha^2 \\ 3\alpha^2 - 3 & 3\alpha & 1 - 9\alpha^2 \\ 3\alpha - 3 & 1 & -3\alpha^2 \end{pmatrix}, C = \begin{pmatrix} -1 + 9\alpha - 18\alpha^2 + 10\alpha^3 \\ 9\alpha - 24\alpha^2 + 15\alpha^3 \\ 3\alpha - 9\alpha^2 + 6\alpha^3 \end{pmatrix}. \qquad (25)$$

As seen from the formula (22), for $n = 3$ the oscillatory component arises in the solution.

The results of calculation are presented in Fig. 3 and 4 for the case $Ct_0 = 1$.

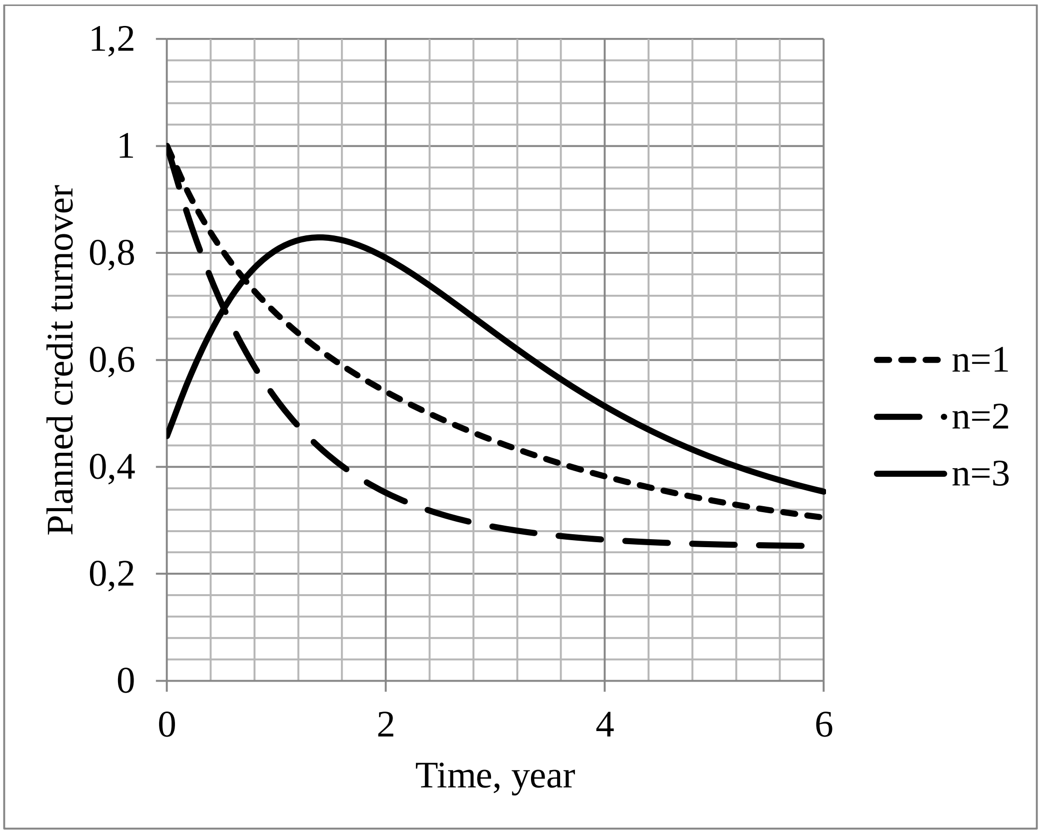


Fig. 3. The program of deposit attraction under changing $\alpha$ from 1 to ¼

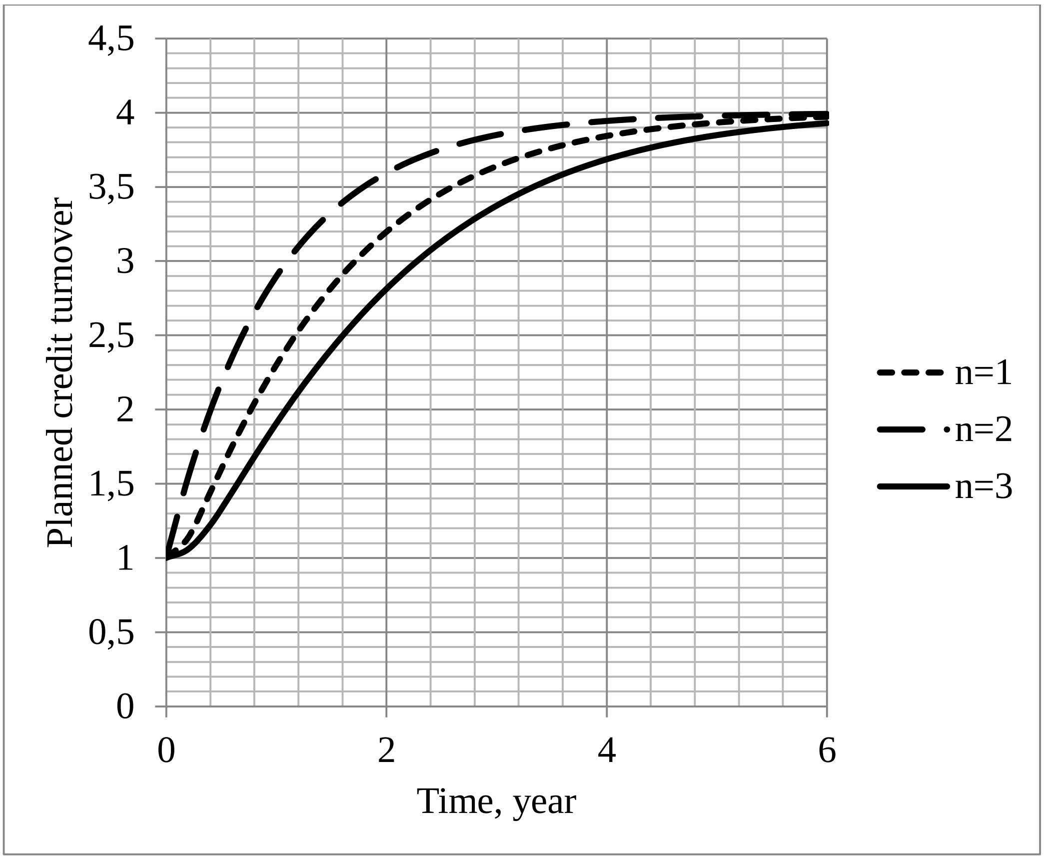


Fig. 4. The program of deposit attraction under changing $\alpha$ from 1 to 4

According to the program of deposit attraction presented in Fig. 3 the average term to maturity at time $t = 0$ increases from $n$ to $4n$. The increase in the average maturity term slows the return of deposits and allow reducting the deposit attraction, i.e. credit turnover, but only for cases when $n = 1$ and 2. Under the distribution terms to maturity with $n = 3$, the program of deposit attraction (credit turnover dynamics) required for supporting the deposit balances at a constant level (see. the condition (3)) becomes unobvious. Bank's deposit division could at first increase in the volume of deposit attracting and then reduce in them.

Under the deposit attraction program shown in Fig. 4, the average term to maturity is reduced from $n$ to $n/4$. Reducing the average maturity speeds return of deposit and requires increasing the deposit attraction, i.e. credit turnover. However, it was found that reducing the average term to maturity for distribution with $n = 3$ requires a growth rate of credit turnovers less than distributions with $n = 1$ and 2.

Thus, the given examples of the deposit attraction programs for maturity distribution with $n$ = 3 show importance of taking into account changes in deposit maturity distribution.

**Summary**

Erlang distribution describes as convex and downward types of distributions that are common for actual deposit maturity distribution. Therefore, such a distribution has convenient analytical representation that allows achieving the closed analytical solution. The explicit solutions for designing the deposit attraction program under changing the deposit maturity distribution are given. Starting with the distribution shape parameter $n$ = 3 and more, oscillatory components arises in the solution.

The deposit attraction program significantly depends on how the maturity distribution changes. Therefore, the effective banks` deposit management is impossible without taking into account this distribution of deposit maturities.

Further research should be directed to the development of discrete deposits models and analyzing their behavior.

Literature


1. BIS (2008). Principles for Sound Liquidity Risk Management and Supervision. http://www.bis.org/publ/bcbs144.pdf.
2. Freedman, B. (2004). An Alternative Approach to Asset-Liability Management. 14-th Annual International AFIR Colloquium (Boston). http://www.actuaries.org/AFIR/Colloquia/Boston/Freedman.pdf.
3. Manzhirov, A.V. (2003). Handbook of Integral Equations. – Moscow: Fizmatlit, 2003. – 608 p.
4. NBU: National Bank of Ukraine (2015). Key Performance Indicators of Banks.

http://www.bank.gov.ua/control/en/publish/article?art_id=37942&cat_id=37937.

5. Prudnikov, A.P., Brychkov, Y.A., Marichev, O.I. (2003). Integrals and Kinds. Elementary Functions. – Moscow: Fizmatlit, 2003. – 623 p.
6. Selyutin, V., Rudenko, M. (2013). Mathematical Model of Banking Firm as Tool for Analysis, Management and Learning. http://ceur-ws.org/Vol-1000/ICTERI-2013-p-401-408-ITER.pdf.
7. Voloshyn, I.V. (2004). Assessment of Banking Risks: New Approaches. – Kyiv: Elha, Nika-Center, 2004. – 216 p.
8. Voloshyn, I.V. (2005). Transitional Dynamics of Liquidity Gaps. National Bank of Ukraine Herald. – 2005 – #9 – P.26-28.
9. Voloshyn, I.V. (2007). The Dynamics of Bank Liquidity Gaps under Variable Program of Allocating and Attracting Funds. National Bank of Ukraine Herald. – 2007. – #8 – P.24-26.
10. Voloshyn, I.V. (2014). An Unobvious Dynamics of Rolled over Time Banking Deposits under a Shift in Depositors' Preferences: *Whether a Decrease in Weighted Average Maturity of Deposits is Indeed an Early Warning Liquidity Indicator?* http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2553732.
11. Wikipedia. Erlang Distribution. http://en.wikipedia.org/wiki/Erlang_distribution.