

Localization properties of solutions for nonlinear parabolic equations

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Let $Q_T = (0, T) \times \Omega$, $0 < T < \infty$, $\Omega \subset \{x \in \mathbb{R}^n : |x| > 1\}$ be a bounded domain in \mathbb{R}^n , $n \geq 1$, with C^1 -boundary $\partial\Omega = \partial_0\Omega \cup \partial_1\Omega$, where $\partial_0\Omega = \{x \in \mathbb{R}^n : |x| = 1\}$, $\partial_1\Omega \subset \{x \in \mathbb{R}^n : |x| > l\}$, $l = \text{const} > 1$. The aim of this brief communication is to investigate the behavior of weak (energy) solutions of the following initial-boundary problem:

$$u_t - \sum_{i=1}^n (a_i(t, x, u, \nabla_x u))_{x_i} + g(t, x)|u|^{q-1}u = 0 \quad \text{in } Q_T, \quad 0 < q < 1; \quad (1)$$

$$u(t, x) = f(t, x) \quad \text{on } (0, T) \times \partial_0\Omega, \quad u(t, x) = 0 \quad \text{on } (0, T) \times \partial_1\Omega; \quad (2)$$

$$u(0, x) = 0 \quad \forall x \in \Omega. \quad (3)$$

Here the functions $a_i(t, x, s, \xi)$ ($i = 1, \dots, n$) are continuous in all arguments and satisfy the following conditions for $(t, x, s, \xi) \in (0, T) \times \Omega \times \mathbb{R}^1 \times \mathbb{R}^n$:

$$|a_i(t, x, s, \xi)| \leq d_1|\xi|, \quad d_1 = \text{const} < \infty,$$

$$\sum_{i=1}^n (a_i(t, x, s, \xi) - a_i(t, x, s, \eta))(\xi_i - \eta_i) \geq d_0|\xi - \eta|^2, \quad d_0 = \text{const} > 0.$$

The absorption potential $g(t, x)$ is continuous nonnegative function such that

$$g(t, x) > 0 \quad \forall (t, x) \in (0, T] \times \bar{\Omega}; \quad g(0, x) = 0 \quad \forall x \in \bar{\Omega}.$$

Following [1], by *energy (weak) solution* of problem (1)–(3) we understand the function $u(t, \cdot) \in f(t, \cdot) + L_2(0, T; H^1(\Omega, \partial\Omega))$ such that $u_t(t, \cdot) \in L_2(0, T; (H^1(\Omega, \partial\Omega))^*)$, and u satisfies (2), (3) and the integral identity:

$$\int_{(0, T)} \langle u_t, \xi \rangle dt + \int_{(0, T) \times \Omega} \sum_{i=1}^n a_i(t, x, u, \nabla_x u) \xi_{x_i} dx dt + \int_{(0, T) \times \Omega} g(t, x)|u|^{q-1}u \xi dx dt = 0$$

$\forall \xi \in L_2(0, T; H^1(\Omega, \partial\Omega))$.

We are interested in a phenomenon called "localization of solutions" for a wide classes of nonlinear parabolic equations with a degenerate absorption potential $g(t, x)$, whose presence plays a significant role for the mentioned nonlinear effect. It is well-known that, if $g(t, x) \geq c_0 > 0 \quad \forall (t, x) \in (0, T] \times \bar{\Omega}$, an arbitrary energy solution of the considered problem has the finite-speed propagation property for solution's support: $\zeta(t) := \sup\{|x| : x \in \text{supp } u(t, \cdot)\} < 1 + c(t)$, where $c(t) \rightarrow 0$ as $t \rightarrow 0$. In particular, this implies *the localization of solution* (see, e. g., [2] and references therein):

$$\zeta(t) := \sup\{|x| : x \in \text{supp } u(t, \cdot)\} < c_1 = c_1(T_1) < l \quad \forall t : 0 \leq t < T_1 = T_1(l) \leq T.$$

For various semi-linear parabolic equations, the localization of solutions' supports were studied by many authors (see, e.g., [3] and references therein). It seems that A. S. Kalashnikov [4] was the first who investigated the localization property for the first initial-boundary problem for a 1-D heat equation. More precisely, he considered problem (1)–(3) in $[1, +\infty)$ with $n = 1$, $a_i(t, x, s, \xi) = \xi \in \mathbb{R}^1$, $g(t, x) = g_0(t) \in C^1([1, +\infty)) \cap L_\infty([1, +\infty))$, $g_0(0) = 0$, $g_0(t) > 0 \forall t > 0$ and $u(t, 1) = f(t) \in C^1([1, +\infty)) \cap L_\infty([1, +\infty))$. Under the assumption $g_0(t)^{-1} \cdot f(t) \rightarrow 0$ as $t \rightarrow 0$, he proved that solutions possess weak localization property for t separated from 0: $\sup\{\zeta(t) : 0 < \delta \leq t < T\} < c_1 = c_1(\delta) < \infty \forall \delta > 0$. On the other hand, following G.I. Barenblatt's conjecture on an initial jump of the free boundary, A. S. Kalashnikov in [4] proved that

$$\inf\{\zeta(t) : 0 < t < t_*\} \geq c_2 = c_2(t_*) > 0, \quad (4)$$

if potential $g_0(t)$ decreases fast enough when $t \rightarrow 0$.

The analysis of [4] concerns only the case of strongly degenerating boundary regimes $f(t)$ (see condition (2)). Our method involutes arbitrary $f(t)$, which are strongly degenerate, weakly degenerate as well as non-degenerate as $t \rightarrow 0$. We found sufficient conditions for the strong localization of solutions (that is continuous propagation of support near to $t = 0$). Note, that these conditions are formulated as a subordination of the boundary regime to the absorption potential. For an arbitrary boundary regime (without any subordination conditions), a certain type of weakened localization is obtained. Under some restriction from below on the degeneration of the potential, the strong localization holds for an arbitrary boundary regime (including regimes that do not satisfy any conditions of subordination).

Our approach is adaptation and combination of a variant of local energy method and an estimate method of Saint-Venant's principle type. These methods are the result of a long evolution of ideas coming from the theory of linear elliptic and parabolic equations. The essence of the energy method consists of special inequalities links different energy norms of solutions. This method was developed and used in [2], [5], [6]. The second approach is a technique of parameter's introduction. This method was offered by G.A. Iosif'jan and O.A. Oleinik [7].

Note, that offered by us combined approach [8] can be applied also to higher order equations.

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