ТЕЗИ ДОПОВІДЕЙ

Чернівці

2025

Example 2. Let $n = 1$. We find the square of δ-function:

$$C(\delta^2)(s) = (C(\delta))^2(s) = \frac{1}{s^2} = \left(\frac{-1}{s}\right)' = (-C(\delta))' = C(-\delta'),$$

i.e.

$$\delta^2 = -\delta'.$$

The ring of formal power series of the form $u(t,x) = \sum_{k=0}^{\infty} u_k(x)t^k$ with coefficients $u_k(x) \in K[x_1, ..., x_n]'$ will be denoted by $K[x_1, ..., x_n]'[[t]]$.

The partial derivative with respect to t of the series $u(t,x) \in K[x_1, ..., x_n]'[[t]]$ is defined by the formula $\frac{\partial u}{\partial t} = \sum_{k=1}^{\infty} ku_k(x)t^{k-1}$. The partial derivatives D^α with respect to variables $x_1, ..., x_n$ of the series $u(t,x) \in K[x_1, ..., x_n]'[[t]]$ is defined as follows: $D^\alpha u(t,x) = \sum_{k=0}^{\infty}(D^\alpha u_k)(x)t^k$.

Theorem 1. *Let $K \supset \mathbb{Q}$ and $a \in K$. Then the Cauchy problem in $K[x_1, \dots, x_n]'[[t]]$:*

$$\frac{\partial u}{\partial t} = au\frac{\partial^n u}{\partial x_1 \cdots \partial x_n}, \quad u(0,x) = \delta(x)$$

has a unique solution. This solution has the form $u(t,x) = \sum_{k=0}^{\infty} u_k \delta^{2k+1}(x)t^k$, where $u_0 = 1$ and $u_k \in K$ satisfy recurrence equation

$$u_{k+1} = (k+1)^{-1}(-1)^n a \sum_{j=0}^{k}(2j+1)^n u_j u_{k-j}, \quad k = 0, 1, 2, \dots$$

Theorem 2. *Let $a \in K$. Then the Cauchy problem in $K[x_1, \dots, x_n]'[[t]]$:*

$$\frac{\partial u}{\partial t} = (-1)^n a \prod_{j=1}^{n} \frac{\partial u}{\partial x_j}, \quad u(0,x) = \delta(x)$$

has a unique solution. This solution has the form $u(t,x) = \sum_{k=0}^{\infty} \frac{A_k(n+1,n^2)}{nk+1} a^k \delta^{nk+1}(x)t^k$, where $A_k(r,m) = \frac{r}{r+mk}\binom{r+mk}{k}$ are Fuss–Catalan–Raney numbers and $\frac{A_k(n+1,n^2)}{nk+1} \in \mathbb{Z}$.

References

[1] S.L. Gefter and A.L. Piven', *Nonlinear Partial Differential Equations in Module of Copolynomials over a Commutative Ring*, J. Math. Phys. Anal. Geom. **21** (2025), No. 3, 319–345.

e-mail: gefter@karazin.ua, aleksei.piven@karazin.ua

Method for Determining the Cyclicity of Singular Points for Quadratic Systems of Differential Equations

Oleksii Haluza, Olena Akhiiezer

National Technical University “Kharkiv Polytechnic Institute”, Kharkiv, Ukraine

Lyudmyla Malyarets, Anatoly Voronin, Stepan Lebedev

Simon Kuznets Kharkiv National University of Economics, Kharkiv, Ukraine

In the theory of nonlinear dynamic systems, which are described using differential equations with polynomial right-hand sides, the identification of hidden periodic regimes (so-called limit cycles) is still relevant. These periodic dynamic trajectories emerge in the vicinity of equilibrium points in a phenomenal way. They are the ones that determine the characteristic features of self-oscillatory behavior, which are inherent only to purely nonlinear objects as a result of the emergence of Andronov-Hopf bifurcations. The methods for studying bifurcations of this type are quite well known for cases where the emerging limit cycle is the only one with a well-defined type of stability [1]. It should be emphasized that the most important problem in the study of Hopf bifurcation is the search for the maximum number of limit cycles that can arise from the equilibrium point with small perturbations of the parameters of the system under study. This problem was completely solved only for the quadratic case of polynomial systems, which were considered by N. N. Bautin's and E. A. Andronova's [2]. They proved that the maximum number of limit cycles that can arise from an equilibrium position (a singular point of the focus type) in objects described by a system of two differential equations with a quadratic part is three. The authors of this study have studied in detail a special case of the Andronova's system, which has two equilibrium points of the focus type, and have proven that three limit cycles arise around each of these foci [3]. However, it should be noted that the Bautin's and Andronova's systems contain only five nonlinear parameters, i.e. these systems are special cases of a general type system in which all six coefficients of the quadratic terms are nonzero. Thus, the search for a rational way to determine the cyclicity of a singular point for a six-parameter system of differential equations with quadratic right-hand sides is relevant.

Let us consider a system of two differential equations containing polynomials of no higher than the second degree:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \alpha x + \gamma y + \alpha_{20}x^2 + \alpha_{11}xy + \alpha_{02}y^2 \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \delta x + \beta y + \beta_{20}x^2 + \beta_{11}xy + \beta_{02}y^2 \end{cases} \tag{1}$$

where all coefficients of quadratic terms are not equal to zero.

It is obvious that system (1) has a trivial equilibrium position $x = 0$, $y = 0$. In the neighborhood of this singular point the characteristic polynomial for system (1) has the following form:

$$\lambda^2 - (\alpha + \beta)\lambda + \alpha\beta - \gamma\delta = 0. \tag{2}$$

Assuming that $\beta = 2\mu - \alpha$, where μ is a small variable, then in this case the characteristic polynomial (2) takes the form:

$$\lambda^2 - 2\mu\lambda - \gamma\delta - \alpha^2 + 2\alpha\mu = 0. \tag{3}$$

If $\mu = 0$, then the characteristic polynomial (3) has imaginary solutions: $\lambda_{1,2} = \pm i\omega$, where $\omega = \sqrt{-\gamma\delta - \alpha^2}$. This means that for this type of equilibrium of the system (1) at the point $(0;0)$ is a complex focus, because when differentiating expression (3) with respect to the parameter we obtain:

$$\left.\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}\right|_{\mu=0} = 1 - \frac{\alpha}{\omega}i. \tag{4}$$

The presence of a complex focus in system (1) gives grounds to assume the existence of one or more limit cycles in the system under consideration.

For further analysis of the problem under study, it is necessary to transform system (1) to the Poincaré normal form. This can be done using variable substitution: $x = \gamma x_1$, $y = -\alpha x_1 - \omega x_2$. Moreover, using the relation $-\gamma\delta - \alpha^2 = 1$ we obtain that $\omega = 1$. This allows us to significantly simplify subsequent computational procedures. Assuming that $\mu = 0$, system (1) can be written as follows:

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = -x_2 + a_{20}\dfrac{x_1^2}{2} + a_{11}x_1x_2 + a_{02}\dfrac{x_2^2}{2} \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = \quad x_1 + b_{20}\dfrac{x_1^2}{2} + b_{11}x_1x_2 + b_{02}\dfrac{x_2^2}{2} \end{cases} \tag{5}$$

where the parameters a_{20}, a_{11}, a_{02}, b_{20}, b_{11}, b_{02} are expressed algebraically in terms of the original parameters α_{20}, α_{11}, α_{02}, β_{20}, β_{11}, β_{02}.

We transform system (5) into a complex differential equation using the variable $z = x_1 + i \cdot x_2$:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = z + g_{20}\frac{z^2}{2} + g_{11}z \cdot \bar{z} + g_{02}\frac{\bar{z}^2}{2}, \tag{6}$$

where $\bar{z} = x_1 - i \cdot x_2$, and the parameters of equation (6) have the following form:

$$\begin{aligned} g_{20} &= 0.25\big(a_{20} - a_{02} + 2b_{11} + i \cdot (b_{20} - b_{02} - 2a_{11})\big), \\ g_{11} &= 0.25\big(a_{20} + a_{02} + i \cdot (b_{20} + b_{02})\big), \\ g_{02} &= 0.25\big(a_{20} - a_{02} - 2b_{11} + i \cdot (b_{20} - b_{02} + 2a_{11})\big). \end{aligned}$$

To determine the maximum limit cycle multiplicity for equation (6), it is necessary to calculate the values of the first three Lyapunov focal quantities. In accordance with the work of H. Zoldek [4], we obtain:

$$\begin{aligned} l_1 &= -\frac{1}{2}\mathrm{Im}(g_{20}g_{11}), \\ l_2 &= -\frac{1}{12}\mathrm{Im}\big((g_{20} - 4\bar{g}_{11})(g_{20} + \bar{g}_{11})\bar{g}_{11}g_{02}\big), \\ l_3 &= -\frac{5}{64}\mathrm{Im}\big((4g_{11}^2 - g_{02}^2)(g_{20} + \bar{g}_{11})\bar{g}_{11}^2 g_{20}\big). \end{aligned} \tag{7}$$

Thus, in accordance with (7), it can be assumed that there is such a type of relationship between the parameters of equation (6):

$$g_{20} = k\bar{g}_{11}, \tag{8}$$

from which it follows that:

1. for a complex value of the coefficient k there is only one limit cycle;
2. for a real value of k (but $k \neq -1$ and $k \neq 4$) there are two limit cycles;
3. if $k = 4$, there are three limit cycles;
4. if $k = -1$, the system is conservative.

Using relations (6) and (8), we obtain parametric conditions for the existence of three limit cycles:

$$\begin{cases} 2b_{11} = 3a_{20} + 5a_{02} \\ 2a_{11} = 5b_{20} + 3b_{02} \end{cases} \tag{9}$$

System (9) describes the dependence of two parameters of the six-parameter quadratic system of differential equations (5) on its other four parameters, which are free. This is a consequence of the fact that the first two Lyapunov quantities are equal to zero. The obtained result confirms the conclusions of N. N. Baunin and E. A. Andronova about the degree of limit cycle multiplicity equal to three for a system of two differential equations with quadratic nonlinearity.

References

[1] Chow S.-N., Li C., Wang D. *Normal Forms and Bifurcation of Planar Vector Fields.* Cambridge: Cambridge University Press, 1994. https://doi.org/10.1017/CBO9780511665639

[2] Gaiko V. *Global Bifurcation Theory and Hilbert's Sixteenth Problem.* Springer Science & Business Media, LLC, 2003. https://doi.org/10.1007/978-1-4419-9168-3

[3] Voronin A., Lebedev S. *Quadratic system of two differential equations with six limit cycles: two approaches to problem.* 2023. https://doi.org/10.13140/RG.2.2.32572.92808

[4] Zoladek H. *Quadratic systems with center and their perturbations*, volume 109(2) of *Journal of Differential Equations.* 1994. P. 223-273. https://doi.org/10.1006/jdeq.1994.1049

e-mail: oleksii.haluza@khpi.edu.ua

Inverse Free Boundary Problems For Degenerate Parabolic Equation

Nadiia Huzyk

Hetman Petro Sahaidachnyi National Army Academy, Lviv, Ukraine

In a free boundary domain $\Omega_T = \{(x,t) : 0 < x < h(t), 0 < t < T\}$, where $h = h(t)$ is an unknown function, it is considered an inverse problems for determination of the time dependent functions $b_1 = b_1(t)$, $b_2 = b_2(t)$ in the minor coefficient in one-dimensional degenerate parabolic equation

$$u_t = t^{\beta} a(t) u_{xx} + (b_1(t)x + b_2(t))u_x + c(x,t)u + f(x,t) \tag{1}$$

with initial condition

$$u(x,0) = \varphi(x), \quad x \in [0, h(0)], \tag{2}$$

boundary conditions

$$u(0,t) = \mu_1(t),\ u(h(t),t) = \mu_2(t),\ t \in [0,T] \tag{3}$$

and overdetermination conditions

$$\int_0^{h(t)} u(x,t)dx = \mu_3(t), \quad t \in [0,T]. \tag{4}$$