INTEGRAL INEQUALITIES IN MACROECONOMIC DYNAMICS

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Annotation. *According to the proposed mathematical model of macroeconomic dynamics, the limit on the value of national income is determined by means of an integral inequality. Due to the inclusion of the dynamic memory factor, this model allows us to take into account the influence of previous states of the system on its current state.*

Keywords: *mathematical model, national income, aggregate profit, dynamic memory factor of the system, Gronwall-Bellman Inequality.*

Анотація. *Згідно із запропонованою математичною моделлю макроекономічної динаміки обмеження на величину національного доходу визначається за допомогою інтегральної нерівності. Завдяки включенню в розгляд фактору динамічної пам'яті ця модель дозволяє враховувати вплив попередніх станів системи на її теперішній стан.*

Ключові слова: *математична модель, національний дохід, сукупний прибуток, фактор динамічної пам'яті системи, нерівність Гронуолла-Беллмана.*

Mathematical modeling is increasingly used to describe economic processes at both micro and macro levels. This is explained by the fact that the use of mathematical models allows us to significantly improve our understanding of the nature of these processes and increase the accuracy of forecasting results. Particular attention is paid to dynamic models, which make it possible to study transition phenomena, assess the limits of stability of the system's equilibrium state, and predict the occurrence of bifurcations and socio-economic cataclysms.

Traditionally, the analysis of macroeconomic systems is based on the well-known relationship:

$$Y = C + I + G, \quad (1)$$

where Y is national income value; C is consumption volume; I is investment volume; G is independent government spending.

In other words, equation (1) characterizes the balance (state of equilibrium) between supply Y and aggregate demand C + I + G.

There are many mathematical models of macroeconomic dynamics, where all elements of equation (1) are functions of time. These models describe the evolution of the system, its transition from some initial state to a stationary one. An example of such dynamic models is a mathematical model that describes the interaction of a «multiplier-accelerator» [1], or a model of an economic stabilization system that involves the synthesis of a certain regulatory policy [2], as well as nonlinear models of economic growth for a system with memory [3] and many others.

In this study, the authors proposed a hypothesis that the peculiarity of the dynamics of macroeconomic processes is such that for any moment in time, supply Y(t) does not exceed aggregate demand C(t) + I(t) + G(t). That is, instead of equation (1), such an inequality takes place:

$$Y(t) \le C(t) + I(t) + G(t), \quad (2)$$

where t is a point in time for which the state of the system is considered.

To specify the structure of inequality (2), the following assumptions are made in the paper. First, we will assume that the volume of investments I(t) is autonomous, i. e. does not depend on the national income value Y(t). In this case, the sum of investment I(t) and independent government spending G(t) is an exogenous variable. Let's designate it as A(t):

$$A(t) = I(t) + G(t). \quad (3)$$

Secondly, we will assume that there is a continuously distributed lag on the consumption side from income. And this lag depends on the previous values of the consumption function G(t):

$$C(t) = c\int_0^t K(t,\tau)Y(\tau)d\tau, \quad (4)$$

where c is the marginal propensity to consume; K(t, τ) is a dynamic memory factor; τ is a moment in time from the period that precedes the moment in question.

Let us introduce into consideration one more parameter s, which describes the marginal propensity to accumulate. The parameters c and s are related to each other by the relation: c + s = 1.

The function K(t, τ) is a function of two variables. It represents the core of the integral expression (4) and has the meaning of a «dynamic memory factor» of the past values of the process under study. Typically, the function K(t, τ) is a decreasing function of time t.

Using relations (2) – (4), we write the basic macroeconomic inequality in explicit form:

$$Y(t) \le c\int_0^t K(t,\tau)Y(\tau)d\tau + A(t). \quad (5)$$

From a mathematical point of view, expression (5) is a linear integral inequality with respect to the income function Y(t). Obviously, the explicit form of the function Y(t) Obviously, the explicit form of the function will depend entirely on the structure of the core K(t, τ). In the case of degeneracy of the core K(t, τ) = α(t) · β(τ), inequality (5) takes the form:

$$Y(t) \le c \cdot \alpha(t)\int_0^t \beta(\tau)Y(\tau)d\tau + A(t). \quad (6)$$

It should be noted that in form (6) the inequality is essentially a special case of the inequality known as Gronwall–Bellman Inequality [4]:

$$Y(t) \le c \cdot \alpha(t)\int_0^t A(\tau)\beta(\tau)\exp\left(c\int_\tau^t \alpha(\xi)\beta(\xi)d\xi\right)d\tau + A(t) \quad (7)$$

where ξ this is a moment in time from the period that belongs to a subsequent moment in time in relation to the moment τ, but preceding the moment in time under consideration t.

For inequalities (5) – (7), we consider an important special case when the core has the following form. Let's assume that $K_1(t,\tau) = \lambda e^{-\lambda(t-\tau)}$, where λ is a parameter that characterizes the rate of «forgetting» of the state of the system at past moments in time and this decrease is described by an exponential dependence. Accordingly, we obtain that $\alpha_1(t) = e^{-\lambda t}$ and $\beta_1(t) = \lambda e^{\lambda t}$. In this case, inequality (6) is transformed as follows:

$$Y(t) \le c \cdot e^{-\lambda t}\int_0^t \lambda e^{\lambda t} Y(\tau)d\tau + A(t). \quad (8)$$

Based on the fact that the function A(t) takes only positive values, and the components of the core $\alpha(t)$ and $\beta(t)$ are non-negative and integrable, we obtain:

$$Y(t) \le c \cdot e^{-\lambda t}\int_0^t A(\tau) \cdot \lambda e^{\lambda t}\exp\left(c\int_\tau^t \lambda d\xi\right)d\tau + A(t)$$

or

$$Y(t) \le c \cdot \lambda\int_0^t A(\tau) \cdot e^{-(1-c)\lambda(t-\tau)}d\tau + A(t). \quad (9)$$

Taking into account that 1 – c = s, we obtain:

$$Y(t) \le c \cdot \lambda\int_0^t A(\tau) \cdot e^{-s\lambda(t-\tau)}d\tau + A(t) \quad (10)$$

If $A(t) = A_0$ =const, we obtain a simple inequality:

$$Y(t) \le \frac{A_0}{s}\left(1 - c \cdot e^{-s\lambda}\right) \quad (11)$$

Inequality (11) demonstrates how the income value evolves from the initial value $Y(t = 0) = A_0$ to the equilibrium value $Y(t \to \infty) = s^{-1} \cdot A_0$. This is a result of the multiplier effect in the Keynesian model. Since $s < 1$, then in this case economic growth takes place.

Thus, the use of a mathematical model of macroeconomic dynamics to describe changes in income over time allows us to estimate the limit of the state's economic potential.

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