

Міністерство освіти і науки України



НАУКОВІ ПРАЦІ
ДОНЕЦЬКОГО НАЦІОНАЛЬНОГО
ТЕХНІЧНОГО УНІВЕРСИТЕТУ

Серія: Інформатика, кібернетика
та обчислювальна техніка

№2(43) 2026



Дрогобич 2026

Публікується згідно з рішенням Вченої ради ДВНЗ «Донецький національний технічний університет» (протокол № 9 від 17.09.2026).

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Збірник зареєстровано в Державному комітеті інформаційної політики, телебачення та радіомовлення України. Свідоцтво: серія КВ, №7374 від 03.06.2003.

Збірник включено до переліку наукових фахових видань України, в яких можуть публікуватися результати дисертаційних робіт на здобуття наукових ступенів доктора наук, кандидата наук та ступеня доктора філософії за спеціальностями:

- F2 Інженерія програмного забезпечення;
- F3 Комп'ютерні науки;
- F5 Кібербезпека та захист інформації;
- F6 Інформаційні системи і технології;
- F7 Комп'ютерна інженерія;
- G7 Автоматизація, комп'ютерно-інтегровані технології та робототехніка.

(наказ Міністерства освіти і науки України №928 від 11 червня 2026 р.)

Збірник "Наукові праці ДонНТУ, серія "Інформатика, кібернетика та обчислювальна техніка" за наказом № 928 МОНУ від 11.06.2026 отримав категорію Б.

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УДК 512.2

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LOCAL INVARIANT COORDINATE SYSTEM OF A REGULAR SURFACE BASED ON PRINCIPAL CURVATURE DIRECTIONS

The paper addresses the problem of forming a local invariant coordinate system for a parametrically defined regular surface. A local orthonormal trihedron is developed with two axes aligned with the principal curvature directions and the third with the surface normal. Based on the first and second fundamental forms, relations for determining the principal directions, direction cosines, and the transition matrix between the Cartesian and local coordinate systems are obtained. The conditions for unambiguous determination of the local frame are established, excluding umbilic points. Four orientation variants of the local trihedron are identified, corresponding to simultaneous reversal of the directions of any two axes and preserving the orientation of space. The resulting frame is determined by the intrinsic geometric characteristics of the surface rather than by the directions of its parametrization, thereby providing geometric invariance with respect to the choice of parameters. Numerical verification using a helicoid and a one-sheet hyperboloid confirms the orthonormality, right-handed orientation, and correspondence of the tangent axes to the principal curvature directions. The proposed approach can be applied to geometric modeling, kinematic surface formation, and analysis of the relative position of geometric objects.

Keywords: local invariant coordinate system, local orthonormal frame, principal curvature directions, principal curvatures, regular surface, fundamental forms, direction cosines, geometric modeling.

DOI: 10.31474/1996-1588-2026-2-43-68-80

Introduction

Modern problems in geometric modeling, computational geometry, CAD/CAM, and surface mechanics require surface geometry to be described in a form independent of the particular parametrization used. For a parametrically defined surface, the directions of its parametric coordinate lines provide a natural coordinate system; however, they are generally not orthogonal and do not have a direct invariant geometric meaning. In contrast, the principal curvature directions are determined by the local geometry of the surface itself and correspond to the principal directions of the Weingarten operator, while the associated principal curvatures characterize its local shape.

This distinction is particularly important for free-form and parametrically defined surfaces, where local shape analysis, curvature-line tracing, geometric approximation, surface mesh generation, positioning, and kinematic modeling rely on geometrically meaningful local directions. Recent studies demonstrate the use of principal directions and curvatures in such problems, including curvature-line tracing on trimmed multipatch surfaces and the analysis and processing of parametric surfaces. The classical Darboux trihedron is traditionally considered along a prescribed curve on a surface. At an arbitrary point of the surface, however, a different problem arises: how to determine a local orthonormal coordinate system directly from the surface geometry,

rather than from the directions of a particular parametrization or curve. Existing approaches mainly concern moving frames along curves or specialized frame constructions. This leaves a need for a local frame at an arbitrary surface point whose tangent axes are determined by the principal curvature directions.

Therefore, the development of a method to transform an arbitrary parametric description of a regular surface into a local orthonormal coordinate system that preserves its invariant geometric characteristics is of practical and theoretical interest. Such a method establishes a direct connection between the parametric representation of a surface and its local geometric structure, providing a basis for geometric modeling and kinematic analysis.

The aim of this study is to develop a method for constructing a local invariant coordinate system for a regular surface at an arbitrary regular point, based on its principal curvature directions, and to investigate its geometric properties and practical applicability.

Subject of research: methods for constructing a local invariant coordinate system of a regular surface based on its principal curvature directions and surface normal, including the determination of the corresponding local orthonormal frame and its orientation with respect to a fixed Cartesian coordinate system.

Literature review

The theoretical foundation of this study is

provided by the differential geometry of surfaces, in which the local geometry of a surface is characterized by the first and second fundamental forms, the surface normal, normal curvature, and principal curvatures. The classical approach to constructing moving geometric frames associated with surfaces and curves on them was established in Darboux's work [5]. This work laid the foundation for the Darboux trihedron, a local orthonormal frame whose orientation is determined by the geometry of a surface and a curve lying on it.

The modern development of the theory of moving frames associated with surfaces and their curves is represented by a number of studies. In particular, [1] considers a modification of the Darboux trihedron and applies it to the construction and investigation of surfaces. In [2], new orthogonal moving frames are introduced for curves lying on a surface and are directly related to the classical Darboux trihedron. The Darboux trihedron is also used in [16, 17] to investigate special classes of surface curves, including geodesics, asymptotic curves, and curvature lines. Further developments of this approach are presented in [8], [13], and [4], where various generalizations and applications of Darboux frames are considered in the study of geometric properties of curves and surfaces.

A related line of research concerns the operator and matrix representations of local surface geometry. In [9], the shape operator and rotation matrices associated with Darboux trihedra on ruled surfaces are investigated. The authors establish relationships between transformations of tangent vectors, the surface normal, and matrix representations of the corresponding geometric operations. These results demonstrate that the local orientation and geometric properties of a surface can be effectively described using matrix transformations.

Of particular relevance to the present study is [19], which addresses the construction of curves and surfaces using the Darboux trihedron. The authors consider the Darboux trihedron as a moving orthonormal frame associated with a surface and determine its unit vectors with respect to a fixed Cartesian coordinate system. Of particular interest in the context of the present work is their treatment of the direction cosines of the trihedron vectors based on the parametric representation of the surface.

The results presented in [19] are closely related to the problem considered in this study, since both approaches involve a local coordinate frame associated with a surface and its orientation with respect to a fixed Cartesian coordinate system. Nevertheless, the geometric principles underlying the construction of the tangent axes are fundamentally different. In [19], the tangent vectors of the Darboux trihedron are determined by the prescribed construction of the moving frame and its relation to the surface and an associated curve. In the present study, by contrast, the

tangent axes are determined directly by the principal curvature directions of the surface. Thus, the principal directions, rather than the parametric coordinate lines or an additionally specified curve, provide the geometric basis for the local frame.

This distinction also determines the role of the surface parametrization in the proposed approach. The parametrization is not regarded as the geometric basis for selecting the tangent axes; instead, it provides the computational representation from which the first and second fundamental forms and the corresponding characteristics of the shape operator are obtained. The local frame is therefore defined by geometric characteristics of the surface that do not depend on the particular choice of its parametrization.

Another important direction of current research concerns the determination and application of principal curvatures and principal curvature directions for parametrically defined surfaces. In [14], the first and second fundamental forms and their relationship with the metric and curvature characteristics of a surface are considered. The study [7] investigates restrictions on the principal curvatures of spline surfaces, using their parametric representations and derivatives to establish global conditions on curvature. In [18], the curvature tensor is computed for surfaces represented by triangular meshes, with principal curvatures and principal curvature directions used as local characteristics of surface geometry.

The practical relevance of principal curvature directions for the analysis of complex parametric and free-form surfaces is demonstrated in [11]. The authors investigate curvature trajectories on trimmed multipatch surfaces and show that curvature lines and other special trajectories can be determined from local curvature characteristics. In [3], the magnitude and direction of principal curvature are likewise used as locally meaningful characteristics in the analysis of surface shape.

Further developments in this field involve describing surface geometry in terms of principal curvatures and coordinate-independent geometric characteristics. In [10], the geometry of principal curvature surfaces is studied using principal curvatures as defining local characteristics. In [15], surface geometry is considered from a coordinate-independent perspective, and moving-frame methods are employed to describe the local kinematics of a surface. These studies further support the transition from a purely parametric description to geometric quantities with invariant meaning.

The literature reviewed above shows that moving frames, Darboux trihedron, principal curvatures, and principal curvature directions have been extensively studied as tools for describing the local geometry of surfaces and curves. However, a specific problem remains insufficiently addressed: the construction of a local orthonormal frame at an arbitrary regular non-umbilic point of a surface in

which the two tangent axes are determined specifically by the principal curvature directions, while the third axis is aligned with the surface normal. Such a construction would provide a local frame determined entirely by the geometric characteristics of the surface, without relying on the orientation of its parametric coordinate lines or on the prior selection of an additional curve on the surface.

Local surface invariant frame construction operator

We formulate the mathematical statement of the problem considered in this study in the most general form. Let

$$S \subset \mathbb{R}^3$$

be a regular parametrically defined surface, and let

$$M \in S$$

be a non-umbilic point of the surface. At the point M , let $\mathbf{n}$ denote the unit normal to the surface, while $\mathbf{e}_1$ and $\mathbf{e}_2$ denote unit vectors along its principal curvature directions.

We introduce the operator

$$F : (S, M) \mapsto R_M,$$

which associates with each pair (S, M) a local orthonormal geometric frame

$$R_M = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\},$$

where

$$\mathbf{e}_3 = \mathbf{n}.$$

The vectors of the local frame satisfy

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \quad i, j = 1, 2, 3,$$

and, for a right-handed orientation,

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3.$$

The tangent vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are determined as the principal directions of the Weingarten operator:

$$W\mathbf{e}_i = k_i \mathbf{e}_i, \quad i = 1, 2,$$

where k_1 and k_2 are the principal curvatures of the surface at M . Since M is a non-umbilic point,

$$k_1 \neq k_2,$$

and the corresponding principal directions are uniquely determined as unoriented lines. Their orientations can then be chosen so that the resulting trihedron has the prescribed orientation.

Thus, the operator can be represented as the mapping

$$F : (S, M) \mapsto (\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}),$$

that assigns to a surface and a point on it a local orthonormal frame oriented according to the invariant geometric characteristics of the surface.

It is important to emphasize that the resulting frame is determined by the local geometry of the surface rather than by the choice of its parametrization. A parametric representation is used only at the computational stage to determine the first and second fundamental forms, the Weingarten operator, and its

principal directions. Therefore, the parametrization is used to compute the frame but does not determine its geometric orientation.

For practical implementation, the construction operator can be represented by the following sequence:

$$S \rightarrow (I, II) \rightarrow W \rightarrow (k_1, \mathbf{e}_1), (k_2, \mathbf{e}_2) \rightarrow R_M,$$

where I, II are the surface fundamental forms for the given parametrization. The next section considers the surface in its classical parametric coordinate representation and develops the corresponding computational form of the proposed operator.

Determining the orientation of the invariant trihedron at a surface point

Unlike the Frenet trihedron, whose axes are geometrically determined independently of the particular parametrization of a curve, defining a corresponding trihedron at a point of a surface involves additional difficulties that require certain assumptions or restrictions. The requirement of mutual orthogonality of the frame axes allows one axis to be assigned to the surface normal, while the other two lie in the tangent plane. Since the directions tangent to the coordinate lines of a surface are generally not mutually perpendicular, one must either restrict consideration to orthogonal coordinate grids or introduce an additional parameter that specifies the orientation of the frame within the tangent plane. In this case, a change in the parametrization of the surface makes the orientation of these axes non-unique.

The requirement of an invariant orientation of the axes therefore calls for a unique determination of the principal curvature directions of the surface at the given point. This condition excludes the umbilic case. In Euclidean three-dimensional space, the plane and the sphere are the two standard surfaces for which every point is umbilic. The resulting formulation is therefore more general, since it does not require any special choice of surface parametrization.

Accordingly, in what follows, we consider a surface-attached frame whose axes form an orthogonal system and are defined by the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, corresponding, respectively, to the two principal curvature directions and the surface normal at the given point. We refer to this frame as the invariant frame of the surface.

Consider a surface parametrically represented in the rectangular Cartesian coordinate system $Oxyz$

$$x = f(u, v), y = \phi(u, v), z = \psi(u, v) \quad (1)$$

and a point M on the surface, specified by the parameter values u_0 and v_0 .

In accordance with the above, the point M is assumed not to be an umbilical point of the surface.

We now proceed to determine the orientation of the axes of the surface-attached trihedron at the point under consideration. For this purpose, we assume that the first and second partial derivatives of the equations

(1), as well as the coefficients E, F, G of the first fundamental form and L, M, N of the second fundamental form obtained from them, are known.

The orientation of the unit surface normal vector $\mathbf{e}_3$ is determined by the direction cosines corresponding to the three angles α_3 , β_3 and γ_3 formed by this vector with the basis vectors i, j and k of the coordinate system $Oxyz$. Representing $\mathbf{e}_3$ as the vector product of the unit vectors tangent to the coordinate lines $u = \text{const}$ and $v = \text{const}$ at this point, we obtain the following expressions for these direction cosines

$$\begin{aligned}\cos \alpha_3 &= \frac{y_u z_v - z_u y_v}{\sqrt{EG - F^2}}, \\ \cos \beta_3 &= \frac{z_u x_v - x_u z_v}{\sqrt{EG - F^2}}, \\ \cos \gamma_3 &= \frac{x_u y_v - y_u x_v}{\sqrt{EG - F^2}},\end{aligned}\quad (2)$$

where $x_u, x_v, y_u, y_v, z_u, z_v$ denote the first partial derivatives of the functions in (1), evaluated at the parameter values u_0 and v_0 .

The principal curvature directions at the given point can be determined in two ways:

- directly from the coefficients of the first and second fundamental forms of the surface;
- from the principal curvatures (or principal radii of curvature) and the coefficients of the fundamental forms.

We first consider the determination of the principal curvature directions at a point of the surface (1) directly from the coefficients of the fundamental forms. We begin with the fundamental expression for the curvature of a normal section at the point:

$$k = \frac{II}{I} = \frac{L du^2 + 2M du dv + N dv^2}{E du^2 + 2F du dv + G dv^2}. \quad (3)$$

The principal curvature directions are obtained from (3) as the directions corresponding to the extremal values of this function with respect to the argument $\frac{du}{dv}$, which leads to the following condition:

$$(LF - ME)du^2 + (LG - NE)dudv + (MG - NF)dv^2 = 0. \quad (4)$$

If the point on the surface is non-umbilic, the coefficients of equation (4) cannot all be zero simultaneously. Moreover, for the roots of this equation to be real, its discriminant must be positive:

$$(LG - NE)^2 + 4(LF - ME)(NF - MG) > 0.$$

Therefore, the only possible case is that two of the coefficients of the squared differentials are simultaneously equal to zero. In this case, the principal curvature directions are determined by the equations $du=0, dv=0$, that is, at the given point, the

coordinate lines coincide with the curvature lines.

Suppose that the first or the third coefficient of this equation is equal to zero. Then equation (4) can be factored into two linear factors, the first of which can be set equal to the differential dv (when $(LF - ME) = 0$) or to the differential du (when $(MG - NF) = 0$). We then obtain two roots of equation (4): the first is, respectively, $dv=0$ (or $du=0$), while the second is given by the ratio of the differentials $\frac{du}{dv}$ (or $\frac{dv}{du}$). Thus, one of the directions of the coordinate lines coincides with a principal curvature direction.

In the remaining cases, including the case when $(LG - NE) = 0$, equation (4) can be divided by the square of one of the differentials. Let this be dv^2 . We then obtain the solution of equation (4) in the form:

$$\begin{aligned}\frac{du}{dv} &= \frac{NE - LG}{2(LF - ME)} \pm \\ &\pm \frac{\sqrt{(NE - LG)^2 + 4(LF - ME)(NF - MG)}}{2(LF - ME)}.\end{aligned}\quad (5)$$

Since equation (4) generally has two roots determined by the ratio of the differentials, let the first of them be denoted by $\frac{du}{dv}$ and the second by $\frac{\delta u}{\delta v}$. To determine the unit vectors of the principal curvature directions of the surface, $\mathbf{e}_1$ and $\mathbf{e}_2$, we express them in terms of the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ to the coordinate lines of surface (1) as follows:

$$\begin{aligned}\mathbf{e}_1 &= \frac{\mathbf{r}_u \frac{du}{dv} + \mathbf{r}_v}{\sqrt{E \left(\frac{du}{dv}\right)^2 + 2F \frac{du}{dv} + G}}, \\ \mathbf{e}_2 &= \frac{\mathbf{r}_u \frac{\delta u}{\delta v} + \mathbf{r}_v}{\sqrt{E \left(\frac{\delta u}{\delta v}\right)^2 + 2F \frac{\delta u}{\delta v} + G}}.\end{aligned}\quad (6)$$

Passing from the vector representation to the coordinate representation, we obtain the direction cosines of these vectors:

$$\begin{aligned}\cos \alpha_1 &= \frac{x_u \frac{du}{dv} + x_v}{\sqrt{E \left(\frac{du}{dv}\right)^2 + 2F \frac{du}{dv} + G}}, \\ \cos \alpha_2 &= \frac{x_u \frac{\delta u}{\delta v} + x_v}{\sqrt{E \left(\frac{\delta u}{\delta v}\right)^2 + 2F \frac{\delta u}{\delta v} + G}};\end{aligned}$$

$$\begin{aligned}\cos \beta_1 &= \frac{y_u \frac{du}{dv} + y_v}{\sqrt{E \left(\frac{du}{dv} \right)^2 + 2F \frac{du}{dv} + G}}, \\ \cos \beta_2 &= \frac{y_u \frac{\delta u}{\delta v} + y_v}{\sqrt{E \left(\frac{\delta u}{\delta v} \right)^2 + 2F \frac{du}{dv} + G}}, \\ \cos \gamma_1 &= \frac{z_u \frac{du}{dv} + z_v}{\sqrt{E \left(\frac{du}{dv} \right)^2 + 2F \frac{du}{dv} + G}}, \\ \cos \gamma_2 &= \frac{z_u \frac{\delta u}{\delta v} + z_v}{\sqrt{E \left(\frac{\delta u}{\delta v} \right)^2 + 2F \frac{du}{dv} + G}}.\end{aligned}\quad (7)$$

where all expressions for the derivatives and the coefficients of the fundamental forms appearing in (5) and (7) are evaluated at the parameter values corresponding to the point M .

We now turn to the second method for determining the principal curvature directions. First, we derive the expressions for the principal curvatures of the surface at the given point. This can be done, for example, using the well-known relations for the Gaussian and mean curvatures of a surface. This leads to a quadratic equation whose roots are:

$$k_{1,2} = \frac{1}{2(EG - F^2)} (LG - 2MF + EN \mp \sqrt{(NE - LG)^2 + 4(LF - ME)(NF - MG)}), \quad (8)$$

where the choice of sign on the right-hand side of (8) is made consistently with the directions given by (5). This can be verified by direct substitution of the expression for du from the right-hand side of (5), together with dv , into (3).

Having determined the principal curvatures, we construct, based on (3), two quadratic forms as follows:

$$A = II - k_1 I, \quad \bar{A} = II - k_2 I. \quad (9)$$

The forms (9) are characterized by a zero discriminant and can therefore be represented as the squares of two linear forms:

$$A = (a_1 du + b_1 dv)^2, \quad \bar{A} = (a_2 du + b_2 dv)^2, \quad (10)$$

where the coefficients a_i, b_i ($i = 1, 2$) can be obtained from (9) in terms of the two principal curvatures and four coefficients of the fundamental forms of the surface, two from the first fundamental form and two from the second. We demonstrate this for the first of the forms (8). To this end, we represent it in the form:

$$A = (L - k_1 E) du^2 + 2(M - k_1 F) du dv + (N - k_1 G) dv^2. \quad (11)$$

The fact that the discriminant of (11) is equal to zero makes it possible to express any one of the coefficients of the quadratic form in terms of the other two. Thus, we obtain three possible representations of a_1 and b_1 :

$$a_1' = \sqrt{L - k_1 E}, \quad a_1'' = \frac{M - k_1 F}{\sqrt{N - k_1 G}}, \quad a_1''' = a_1';$$

$$b_1' = \frac{M - k_1 F}{\sqrt{L - k_1 E}}, \quad b_1'' = \sqrt{N - k_1 G}, \quad b_1''' = b_1''. \quad (12)$$

According to (3), setting the first of the forms (9) equal to zero determines the direction corresponding to the first principal curvature of the surface at the point M . We obtain this direction in the form:

$$\frac{du}{dv} = -\frac{b_1}{a_1}, \quad (13)$$

where a_1 and b_1 are determined by formulas (12).

In an analogous manner, the second principal curvature direction can be determined from the coefficients of the second of the forms (10):

$$\frac{\delta u}{\delta v} = -\frac{b_2}{a_2}, \quad (14)$$

where the expressions for a_2 and b_2 are obtained from (12) by replacing k_1 with k_2 .

Eliminating the coefficients of the differentials from the discriminants of the forms (9) also makes it possible to express two coefficients of the quadratic forms of the surface—one from the first fundamental form and one from the second—in terms of the remaining four coefficients and the principal curvature functions. For example, the expressions a_1', b_1' and a_2', b_2' eliminate the corresponding coefficients $N - k_1 G$ and $N - k_2 G$. In this case, the coefficients N and G can be represented in the following form:

$$\begin{aligned}N &= \frac{k_1}{(k_1 - k_2)} b_2'^2 + \frac{k_2}{(k_2 - k_1)} b_1'^2, \\ G &= \frac{b_2'^2}{(k_1 - k_2)} + \frac{b_1'^2}{(k_2 - k_1)}.\end{aligned}$$

Similarly, in the second representation (a_1'', b_1'' and a_2'', b_2''), the coefficients L and E can be expressed in terms of the coefficients M, N, F, G and the principal curvatures. In the third representation, the middle coefficients of the quadratic forms can be expressed in terms of the first and third coefficients, as well as k_1 and k_2 .

We determine the unit vectors of the principal

curvature directions $\mathbf{e}_1$ and $\mathbf{e}_2$ according to (6) by substituting the right-hand sides of (13) and (14) into the numerator, together with the expression for one of the coefficients of the first fundamental form of the surface obtained above. After simplification, the result can be written in the following form:

$$\mathbf{e}_1 = n_1 \mathbf{r}_u + n_2 \mathbf{r}_v, \quad \mathbf{e}_2 = \bar{n}_1 \mathbf{r}_u + \bar{n}_2 \mathbf{r}_v, \quad (15)$$

where the coefficients n_1, n_2 and $\bar{n}_1, \bar{n}_2$ are given by the corresponding expressions for the representations in (12):

– for the first representation

$$n_1' = \frac{M - k_1 F}{(LF - ME)} \sqrt{\frac{L - k_2 E}{k_1 - k_2}},$$

$$n_2' = \frac{L - k_1 E}{(ME - LF)} \sqrt{\frac{L - k_2 E}{k_1 - k_2}},$$

$$\bar{n}_1' = \frac{M - k_2 F}{(LF - ME)} \sqrt{\frac{L - k_1 E}{k_2 - k_1}},$$

$$\bar{n}_2' = \frac{L - k_2 E}{(ME - LF)} \sqrt{\frac{L - k_1 E}{k_2 - k_1}}.$$

– for the second representation

$$n_1'' = \frac{N - k_1 G}{(MG - NF)} \sqrt{\frac{N - k_2 G}{k_1 - k_2}},$$

$$n_2'' = \frac{M - k_1 F}{(NF - MG)} \sqrt{\frac{N - k_2 G}{k_1 - k_2}},$$

$$\bar{n}_1'' = \frac{N - k_2 F}{(MG - NF)} \sqrt{\frac{N - k_1 G}{k_2 - k_1}},$$

$$\bar{n}_2'' = \frac{M - k_2 F}{(NF - MG)} \sqrt{\frac{N - k_1 G}{k_2 - k_1}}.$$

– for the third representation

$$n_1''' = \frac{1}{(LG - NE)} ((N - k_1 G) \sqrt{\frac{L - k_2 E}{k_1 - k_2}} + (M - k_1 F) \sqrt{\frac{N - k_2 G}{k_1 - k_2}}),$$

$$n_2''' = \frac{1}{(NE - LG)} ((L - k_1 E) \sqrt{\frac{N - k_2 G}{k_1 - k_2}} + (M - k_1 F) \sqrt{\frac{L - k_2 E}{k_1 - k_2}});$$

$$\bar{n}_1''' = \frac{1}{(LG - NE)} ((N - k_2 G) \sqrt{\frac{L - k_1 E}{k_2 - k_1}} + (M - k_2 F) \sqrt{\frac{N - k_1 G}{k_2 - k_1}}), \quad (16)$$

$$\bar{n}_2''' = \frac{1}{(NE - LG)} ((L - k_2 E) \sqrt{\frac{N - k_1 G}{k_2 - k_1}} + (M - k_2 F) \sqrt{\frac{L - k_1 E}{k_2 - k_1}});$$

By comparing the denominators in equations (16) with the coefficients in (4), and taking into account the remarks concerning these coefficients, we can conclude that, in the general case, when the tangents to the coordinate lines do not coincide with the principal curvature directions, any of the three representations can be used. If one of the principal curvature directions at the given point coincides with the tangent to the coordinate line $u = \text{const}$, the first representation can be used; conversely, the second representation should be used when the tangent to the $v = \text{const}$ coincides with a principal curvature direction. The third representation can be used in both cases.

When both principal curvature directions coincide with the directions of the tangents to the coordinate lines, the coefficients in equations (15) take the following form:

$$n_1 = 0, n_2 = \frac{1}{\sqrt{G}}, \quad \bar{n}_1 = \frac{1}{\sqrt{E}}, \quad \bar{n}_2 = 0, \\ \left(k_1 = \frac{N}{G}, k_2 = \frac{L}{E} \right).$$

For the direction cosines of the principal curvature directions, we obtain:

$$\cos \alpha_1 = n_1 x_u + n_2 x_v, \quad \cos \alpha_2 = \bar{n}_1 x_u + \bar{n}_2 x_v;$$

$$\cos \beta_1 = n_1 y_u + n_2 y_v, \quad \cos \beta_2 = \bar{n}_1 y_u + \bar{n}_2 y_v;$$

$$\cos \gamma_1 = n_1 z_u + n_2 z_v, \quad \cos \gamma_2 = \bar{n}_1 z_u + \bar{n}_2 z_v,$$

where the parameter values used to compute the coefficients in (16) correspond to the point M .

Having obtained the expressions for the direction cosines of the trihedron axes, we define the axes of the moving Cartesian coordinate system to coincide with the directions of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ of the trihedron at the point $M(x_0, y_0, z_0)$. We then obtain the standard coordinate transformation formulas:

$$x = x_0 + \cos \alpha_1 x' + \cos \alpha_2 y' + \cos \alpha_3 z',$$

$$y = y_0 + \cos \beta_1 x' + \cos \beta_2 y' + \cos \beta_3 z',$$

$$z = z_0 + \cos \gamma_1 x' + \cos \gamma_2 y' + \cos \gamma_3 z'.$$

The inverse transformation formulas:

$$x' = (x - x_0) \cos \alpha_1 + (y - y_0) \cos \beta_1 + (z - z_0) \cos \gamma_1,$$

$$y' = (x - x_0) \cos \alpha_2 + (y - y_0) \cos \beta_2 + (z - z_0) \cos \gamma_2, \quad (17)$$

$$z' = (x - x_0) \cos \alpha_3 + (y - y_0) \cos \beta_3 + (z - z_0) \cos \gamma_3.$$

These formulas can also be written in matrix form:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos \alpha_1 & \cos \beta_1 & \cos \gamma_1 \\ \cos \alpha_2 & \cos \beta_2 & \cos \gamma_2 \\ \cos \alpha_3 & \cos \beta_3 & \cos \gamma_3 \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix}. \quad (18)$$

Before applying the coordinate transformation formulas, the axes of the trihedron must be oriented consistently with the orientation of space, either as a right-handed or a left-handed triple of vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. The presence of square roots means that the direction cosines of the trihedron unit vectors can be determined only up to sign. On the other hand, according to the formulas of analytic geometry, these quantities are not independent. Therefore, the following approach can be used: first, determine any two vectors from the triple $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, and then define the direction cosines of the third unit vector as the coordinates of the cross product of the two vectors already obtained. For example, let us determine the direction cosines of $\mathbf{e}_1$ and $\mathbf{e}_2$ using (16). Then, for $\mathbf{e}_3$, we obtain:

$$\begin{aligned} \cos \alpha_3 &= \cos \beta_1 \cos \gamma_2 - \cos \gamma_1 \cos \beta_2, \\ \cos \beta_3 &= \cos \gamma_1 \cos \alpha_2 - \cos \alpha_1 \cos \gamma_2, \\ \cos \gamma_3 &= \cos \alpha_1 \cos \beta_2 - \cos \beta_1 \cos \alpha_2. \end{aligned}$$

The transition from the parametric equations of the surface (1) to the equations with respect to the coordinate system associated with the moving frame is performed using (17), by substituting the right-hand sides of (1) for the original coordinates x , y , and z . Accordingly, four possible configurations of the surface relative to the tangent plane and its principal curvature directions should be considered.

Numerical verification of the local invariant coordinate system

To numerically verify the proposed method for constructing a local invariant coordinate system, studies were conducted on regular surfaces of different types. The computations were performed in the Julia 1.10.2 environment using the LinearAlgebra and GLMakie packages. The LinearAlgebra package was used for vector and matrix operations, computation of dot and cross products and vector norms, solution of systems of linear equations, and verification of the orthonormality and orientation of the constructed frame. The GLMakie package was used to visualize the surfaces under study and the corresponding local coordinate systems. The source code is available in the accompanying Zenodo repository [6].

In each experiment, for the given surface parametrization and the selected regular point, the tangent vectors, the coefficients of the first and second fundamental forms, the unit normal, the principal curvatures, and the corresponding principal curvature directions were computed. In addition, the consistency of the obtained directions with the eigenvectors of the generalized eigenvalue problem

$$b\xi = k g\xi.$$

where $g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ and $b = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$ are the matrices of the first and second fundamental forms, respectively, k is the principal curvature, and $\xi = (A, B)^T$ contains the coordinates of the corresponding direction in the parametric basis $\{\mathbf{r}_u, \mathbf{r}_v\}$. The corresponding spatial direction on the surface was then determined by the relation:

$$\mathbf{e} = A\mathbf{r}_u + B\mathbf{r}_v.$$

The agreement between the normal curvatures in these directions and the corresponding principal curvatures was also verified:

$$\kappa_n(\xi) = \frac{\xi^T b \xi}{\xi^T g \xi}.$$

Based on the directions obtained, a local orthonormal frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, was constructed.

The correctness of the constructed frame was evaluated according to the following conditions:

$$Q^T Q = I, \quad \det Q = 1, \quad \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3,$$

where $Q = [\mathbf{e}_1; \mathbf{e}_2; \mathbf{e}_3]$.

Four symmetric configurations of the surface with respect to the local frame were additionally considered. They differ in that they simultaneously reverse any two frame axes while preserving the orientation of space. The principal curvature directions, considered as unoriented lines, and the surface normal line remain unchanged.

The numerical verification of the construction was considered successful when the above geometric relations were satisfied to machine-precision accuracy.

Numerical experiment on a right helicoid

The first experiment was conducted for a right helicoid parametrically defined by the equations

$$\mathbf{r}(u, v) = \begin{pmatrix} u \cos v \\ u \sin v \\ cv \end{pmatrix},$$

where $c=0.8$.

As the test point on the surface, we selected the point corresponding to the parameter values

$$u_0 = 1, \quad v_0 = 0.$$

Its Cartesian coordinates are given by

$$P_0 = \mathbf{r}(u_0, v_0) = (1, 0, 0).$$

At this point, the tangent direction vectors were computed as

$$\mathbf{r}_u = \begin{pmatrix} \cos v \\ \sin v \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{r}_v = \begin{pmatrix} -u \sin v \\ u \cos v \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0.8 \end{pmatrix}.$$

The coefficients of the first fundamental form of this surface, for the selected point

$$E = 1, \quad F = 0, \quad G = u^2 + c^2 = 1.64.$$

The unit normal to the surface at the selected point was computed as

$$\mathbf{n} = (0, -0.62469505, 0.78086881).$$

For the second fundamental form at the test point, we obtain

$$L = 0, \quad M = -\frac{c}{\sqrt{u^2 + c^2}} = -0.6246950478, \quad N = 0.$$

An analysis of the coefficients of the fundamental forms shows that the coordinate grid on the helicoid is orthogonal but does not coincide with the curvature lines. In this parametrization of the right helicoid, the principal directions bisect the angles between the coordinate lines on the surface.

The principal curvatures of the surface were then determined from the coefficients of the first and second fundamental forms. Their numerical values are

$$k_1 = -0.4878048780, \quad k_2 = 0.4878048780.$$

Since the principal curvatures are distinct, the selected point is non-umbilic, and therefore the principal curvature directions at this point are uniquely determined as unoriented lines.

The principal curvature directions were determined using the coefficients of the fundamental forms. After normalizing the corresponding tangent direction vectors, we obtain

$$\mathbf{e}_1 = (0.70710678, -0.55215763, -0.44172610),$$

$$\mathbf{e}_2 = (0.70710678, 0.55215763, 0.44172610).$$

The main numerical verification results for the local frame are presented in Table 1.

Table 1 – Numerical verification results for the constructed local frame of a helicoid

Verification	Result
$\ \mathbf{e}_1\ $	1.000000
$\ \mathbf{e}_2\ $	1.000000
$\ \mathbf{e}_3\ $	1.000000
$\mathbf{e}_1 \cdot \mathbf{e}_2$	$\approx 1.0 \cdot 10^{-16}$
$\mathbf{e}_1 \cdot \mathbf{e}_3$	$\approx 1.0 \cdot 10^{-16}$
$\mathbf{e}_2 \cdot \mathbf{e}_3$	$\approx 1.0 \cdot 10^{-16}$
$Q^T Q = I$	confirmed
$\det Q = 1$	1.000000
$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$	confirmed
$\mathbf{e}_3 \cdot \mathbf{n}$	1.000000
$\mathbf{e}_1$ and $\mathbf{e}_2$ belongs to the tangent plane	confirmed
$k_n(\mathbf{e}_1)$	$0.4878048780 = k_2$
$k_n(\mathbf{e}_2)$	$-0.4878048780 = k_1$

The four variants of the symmetric arrangement of the local frame were additionally verified. The results are presented in Table 2. For visual assessment, the corresponding configurations are shown in Fig. 1.

The results presented in Table 2 confirm that all four variants are orthonormal right-handed frames associated with the same principal curvature directions and surface normal line. They differ only in the orientation of the individual frame axes. Thus, the numerical experiment on the helicoid confirms the correctness of the proposed method for constructing the local invariant coordinate system and demonstrates that it admits four orientation variants.

Table 2 – Numerical verification results for the four variants of the local frame for the helicoid example

Verification	Case 1	Case 2	Case 3	Case 4
$\ f_1\ = \ f_2\ = \ f_3\ $	✓	✓	✓	✓
Orthonormality $Q^T Q = I$	✓	✓	✓	✓
$\det Q = 1$	1.0000	1.000	1.000	1.000
$f_1 \times f_2 = f_3$	✓	✓	✓	✓
f_1 — principal curvature direction	✓	✓	✓	✓
f_2 — principal curvature direction	✓	✓	✓	✓
f_3 — surface normal line	✓	✓	✓	✓
Orientation of f_1 relative to $\mathbf{e}_1$	$+\mathbf{e}_1$	$-\mathbf{e}_1$	$+\mathbf{e}_1$	$-\mathbf{e}_1$
Orientation of f_2 relative to $\mathbf{e}_2$	$+\mathbf{e}_2$	$-\mathbf{e}_2$	$-\mathbf{e}_2$	$+\mathbf{e}_2$
Orientation of f_3 relative to $\mathbf{e}_3$	$+\mathbf{e}_3$	$+\mathbf{e}_3$	$-\mathbf{e}_3$	$-\mathbf{e}_3$

Numerical Experiment on a One-Sheet Hyperboloid

The second numerical experiment was conducted for a one-sheet hyperboloid parametrically defined by the following equations:

$$\mathbf{r}(u, v) = \begin{pmatrix} -a \sin u + av \cos u \\ a \cos u + av \sin u \\ cv \end{pmatrix},$$

for $a=1$ and $c=0.8$.

As the test regular point on the surface, we selected the point corresponding to the parameter values $u_0 = 0.3$ and $v_0 = 0.7$, whose Cartesian coordinates in the global coordinate system are:

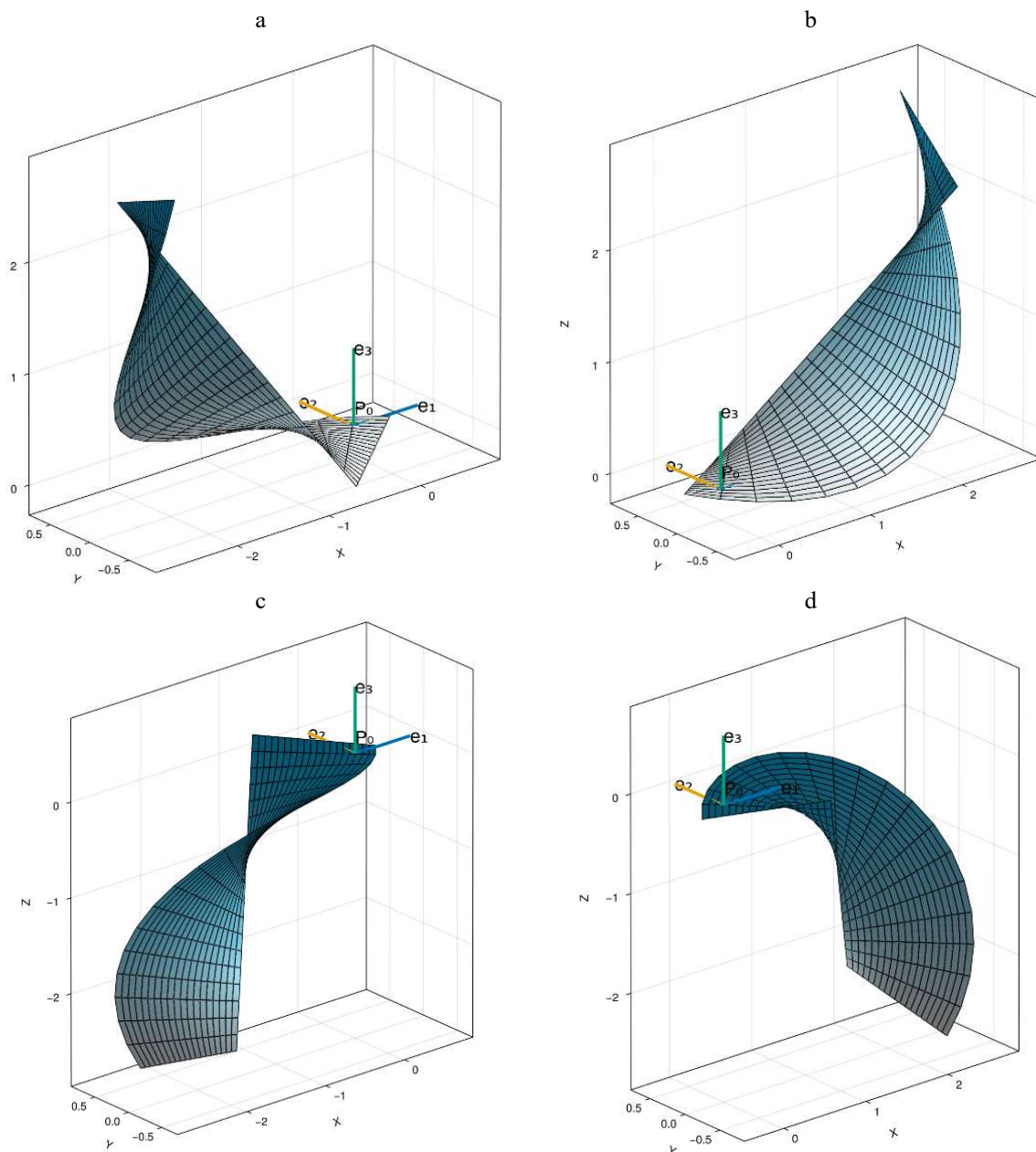


Figure 1 – Visualization of the results for the four configurations of the right helicoid with respect to the local invariant frame: (a) Case 1, (b) Case 2, (c) Case 3, (d) Case 4.

At this point, the coefficients of the first and second fundamental forms are given by:

$$E = 1.49, \quad F = -1, \quad G = 1.64, \\ L = -0.9920939900, \quad M = 0.6658348926, \quad N = 0.$$

Analyzing the coefficient values, we can conclude that the coordinate grid on the surface is not orthogonal in this parametrization, and none of the directions tangent to the coordinate lines coincides with a principal curvature direction.

The numerical values of the principal curvatures determined at the test point are:

$$k_1 = -0.6658348926, \quad k_2 = 0.4612322614.$$

Based on these principal curvatures, the local frame $\{e_1, e_2, e_3\}$ was constructed. The resulting unit vectors of the local frame are

$$e_1 = (0.1781315923, 0.5547056342, 0.8127550647), \\ e_2 = (-0.9521118572, 0.3057499163, -8.8735 \times 10^{-17}), \\ e_3 = (0.2484997930, 0.7738337342, -0.5826055310).$$

The main numerical verification results for the constructed local frame are presented in Table 3.

Table 3 – Numerical verification results for the local frame of the one-sheet hyperboloid

Verification	Result
$\ \mathbf{e}_1\ $	1.000000
$\ \mathbf{e}_2\ $	1.000000
$\ \mathbf{e}_3\ $	1.000000
$\mathbf{e}_1 \cdot \mathbf{e}_2$	$-9.44 \cdot 10^{-17}$
$\mathbf{e}_1 \cdot \mathbf{e}_3$	$-2.22 \cdot 10^{-16}$
$\mathbf{e}_2 \cdot \mathbf{e}_3$	$3.81 \cdot 10^{-18}$
$\mathbf{Q}^T \mathbf{Q} = \mathbf{I}$	confirmed
$\det \mathbf{Q} = 1$	1.000000
$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$	confirmed
$\mathbf{e}_3 \cdot \mathbf{n}$	1.000000
$\mathbf{e}_1$ and $\mathbf{e}_2$ belongs to the tangent plane	confirmed
$\kappa_n(\mathbf{e}_1)$	$0.4612322614 = k_2$
$\kappa_n(\mathbf{e}_2)$	$-0.6658348926 = k_1$

Thus, the resulting vectors form an orthonormal frame, since their norms are equal to one and their pairwise dot products are of the order of machine precision. Verification of the geometric meaning of the axes showed that $\mathbf{e}_3$ is the unit normal to the surface, whereas $\mathbf{e}_1$ and $\mathbf{e}_2$ lie in the tangent plane. The errors in verifying the principal curvature directions are within machine precision.

As in the example of the right helicoid, four variants of the symmetric arrangement of the local frame were also verified. The results are analogous to those presented in Table 2.

A visualization of the configurations of the one-sheet hyperboloid with respect to the axes of the local invariant coordinate system is shown in Fig. 2.

Conclusions

The novelty of the results lies in the development of a method for constructing a local invariant orthonormal coordinate system at an arbitrary regular non-umbilic point of a parametrically defined surface, in which the tangent axes are aligned with the principal curvature directions, and the third axis is aligned with the surface normal. This provides a geometrically invariant representation of the local surface orientation independent of the choice of

parametrization and establishes a basis for its application in geometric modeling and kinematic surface formation.

Based on the first and second fundamental forms, relations for determining the direction cosines of the local frame axes and the transition matrix between the local and fixed Cartesian coordinate systems have been obtained. The condition for the unambiguous determination of the tangent axes in terms of the inequality of the principal curvatures has been established, and four orientation variants of the local frame that preserve the orientation of space have been identified.

The practical significance of the results lies in providing a local frame of reference whose orientation is determined by the geometric characteristics of the surface rather than by the directions of its parametric coordinates. This makes it possible to use the proposed frame for local geometric analysis, geometric modeling, kinematic surface formation, and investigation of the relative positions of geometric objects. The obtained expressions for the direction cosines and the transition matrix provide a direct basis for the practical implementation of this local geometric description.

Further research should investigate the possibility of extending the proposed method to other surface representations. In particular, its application to smooth patches of piecewise-smooth spline surfaces and to surfaces represented by polygonal meshes is of interest. For such representations, the conditions of regularity and smoothness, methods for determining principal curvatures and principal curvature directions, and the conditions for the consistent construction of the local invariant frame within individual patches and across their boundaries require further investigation.

Use of AI and AI-based technologies

During the preparation of the manuscript, ChatGPT was used to check grammar and style and to assist in searching for and structuring relevant scientific literature. AI was not involved in the interpretation of scientific data, formulation of conclusions, or development of the theoretical basis of the study. The author bears full responsibility for the final text of the article, the results obtained, and their scientific interpretation.

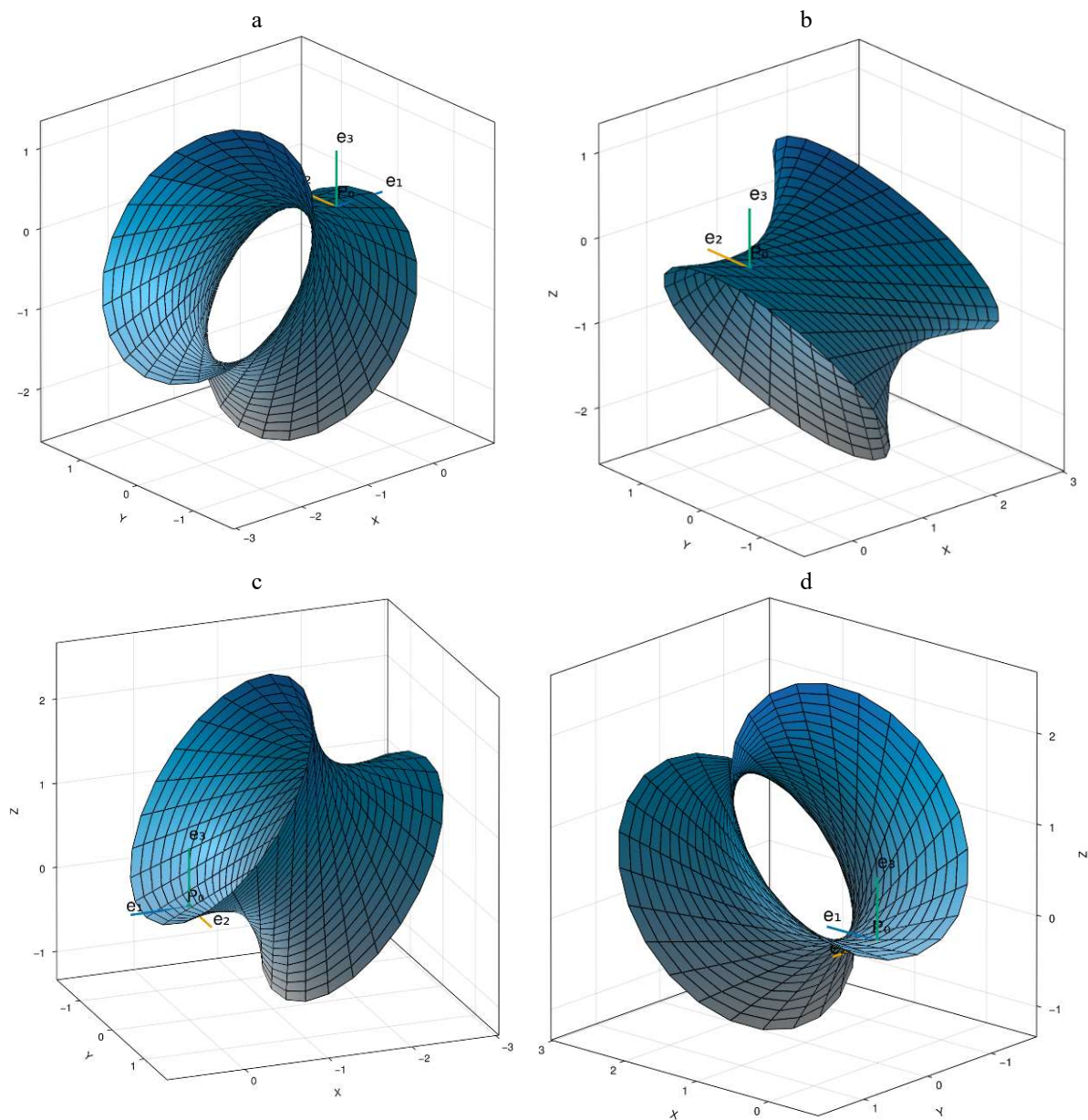


Figure 2 – Visualization of the results for the four configurations of the one-sheet hyperboloid with respect to the local invariant frame: (a) Case 1, (b) Case 2, (c) Case 3, (d) Case 4

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Надійшла до редакції 10.09.2026

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ЛОКАЛЬНА ІНВАРІАНТНА СИСТЕМА КООРДИНАТ РЕГУЛЯРНОЇ ПОВЕРХНІ ЗА ГОЛОВНИМИ НАПРЯМКАМИ КРИВИЗНИ

Розглянуто задачу побудови локальної системи координат для параметрично заданої регулярної поверхні, орієнтованої відповідно до її інваріантних геометричних характеристик у заданій точці. Обґрунтовано доцільність використання рухомого репера, пов'язаного з нормаллю до поверхні та головними напрямками кривизни, оскільки напрямки координатних ліній параметризації поверхні загалом не є ортогональними та не визначаються виключно її геометрією. На основі першої та другої фундаментальних форм поверхні визначено рівняння для знаходження головних напрямків кривизни та відповідних одиничних векторів. Побудовано локальний ортонормований репер, дві осі якого орієнтовані вздовж головних напрямків кривизни, а третя збігається з одиничною нормаллю поверхні. Отримано співвідношення для визначення напрямних косинусів побудованого репера та матрицю переходу від початкової декартової системи координат до локальної системи, пов'язаної з геометрією поверхні. Визначено умови однозначності побудови такої системи, зокрема, необхідність виключення омбілічних точок, у яких головні напрямки кривизни не визначаються однозначно. Запропонований підхід забезпечує побудову локальної системи координат, геометрично пов'язаної з поверхнею та незалежної від вибору її параметризації, і може бути використаний у задачах геометричного моделювання, кінематичного формування поверхонь і дослідження взаємного положення геометричних об'єктів.

Ключові слова: параметрично задана поверхня, головні напрямки кривизни, головні кривизни, фундаментальні форми поверхні, нормаль до поверхні, локальна система координат, напрямні косинуси.